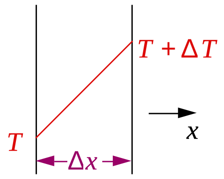
5. MELTING

5.1



1. Preliminaries

a) If heat is transferred in a solid or a stationary fluid

$$Q = -k\frac{dT}{dx}$$

heat flux

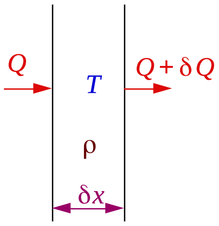
(cal cm^{-2} s^{-1})

proportional to temperature gradient

thermal conductivity (cal cm^{-1} s^{-1} K^{-1})

heat is transferred *down* the temperature gradient from hot to cold

b) If a different amount of heat flows into a region than emerges from it, the region must change its temperature



$$\rho c_p \delta T \delta x = -\delta Q \delta t$$

specific heat / unit mass

$$\rho c_p \frac{\partial T}{\partial t} = -\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right) = k\frac{\partial^2 T}{\partial x^2}$$

if k is a constant

Alternatively,

5.2

$k = \rho c_p \kappa$ ← thermal diffusivity (cm^2 s^{-1}) (c.f. viscosity)

Then

$$T_t = \kappa T_{xx}$$

c) In more than one dimension $\mathbf{Q} = -k\nabla T$

could be anisotropic

$$T_t = \kappa \nabla^2 T$$

Temperature diffuses through the solid

d) Differences in heat flux at an interface can be taken up by a phase change

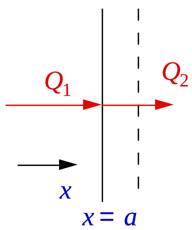
latent heat per unit mass

$$\rho_s L\dot{a} = L_V\dot{a} = Q_1 - Q_2$$

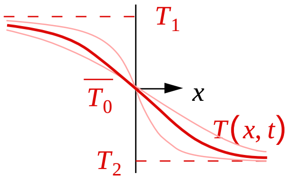
latent heat per unit volume

$$= k\frac{\partial T_2}{\partial x} - k\frac{\partial T_1}{\partial x} \qquad (x = a)$$

$$\text{or} \quad = H - k\frac{\partial T_1}{\partial x} \qquad (x = a)$$



2. Consider 2 semi-infinite bodies of identical physical properties in contact at initial temperatures T_1 and T_2. If heat transfer takes place solely by conduction, and there is NO melting, the governing equations are 5.3



$$T_t = \kappa T_{xx} \qquad (x \gtrless 0) \qquad \text{I}$$

$$\left.\begin{aligned} T &= T_1 \quad (x<0) \\ &= T_2 \quad (x>0) \end{aligned}\right\} [t=0] \qquad \text{II}$$

T and all derivatives continuous at $x = 0,\ t > 0$ III

Because there is no length scale the solution must be in terms of the similarity variable
$$\eta = \frac{x}{2\sqrt{\kappa t}}$$

The differential equation then becomes the ordinary differential equation
$$\frac{d^2T}{d\eta^2} - \eta \frac{dT}{d\eta} = 0$$

with solution
$$T(x,t) = \underbrace{\tfrac{1}{2}(T_1+T_2)}_{T(0,t>0) = \bar{T}_0} + \tfrac{1}{2}(T_1-T_2)\,\mathrm{erf}\left(\eta = \frac{x}{2\sqrt{\kappa t}}\right)$$

where $\mathrm{erf}(x) = \dfrac{2}{\sqrt{\pi}} \displaystyle\int_0^x e^{-t^2}\,dt$

3. Consider now region 1 to be fluid and region 2 to be solid. 5.4
If $\bar{T}_0$ exceeds the melting temperature of the solid, or is less than the solidification temperature of the liquid, a phase change will occur and the boundary at $x = s$ between fluid and solid will migrate. The equations are I and II plus

$$T = T_*, \quad L_V \dot{s} = kT_x(s+) - kT_x(s-) \qquad \text{IV}$$

Again, because there is no external length scale, there is a similarity solution in terms of

$$\eta = \frac{x}{2(\kappa t)^{\frac{1}{2}}} \quad \text{and} \quad s(t) = 2\lambda(\kappa t)^{\frac{1}{2}}$$

$$T(x,t) = T_1 - (T_1 - T_*)\frac{1+\mathrm{erf}\,\eta}{1+\mathrm{erf}\,\lambda} \qquad (\eta < \lambda,\ x < s)$$

$$= T_2 - (T_2 - T_*)\frac{1-\mathrm{erf}\,\eta}{1-\mathrm{erf}\,\lambda} \qquad (\eta > \lambda,\ x > s)$$

with
$$\frac{1}{1+\mathrm{erf}\,\lambda} - \left(\frac{T_* - T_2}{T_1 - T_*}\right)\frac{1}{1-\mathrm{erf}\,\lambda} = \frac{\lambda L_V \pi^{\frac{1}{2}}}{c(T_1 - T_*)}$$

$(k = c\kappa)$

specific heat / *unit volume*

5.5

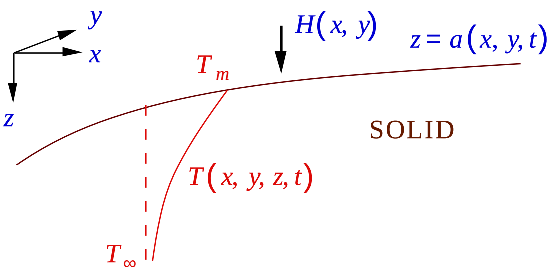
4. There are many extensions possible that consider only conductive transfers (see, for example, books by Carslaw and Jaeger, Crank and Hill).

Consideration of dissolving would be very similar, except that composition C substitutes for temperature T and diffusion co-efficient in solid is so small that it can be set to zero. Could also consider two-component melting and dissolving, which is generally similar to two-component solidification (with sign of boundary growth rate changed).

Convective motions, however, can make considerable alterations because thermal transfers can be so much larger.

5. Consider the following basic problem

5.6



equations

$T_t = \kappa \nabla^2 T \quad (z > a) \qquad T(x, y, a) = T_m \qquad T \to T_\infty \ (z \to \infty)$

long-term solution if $l_x, l_y >> l_z = \kappa / \dot{a}$ is

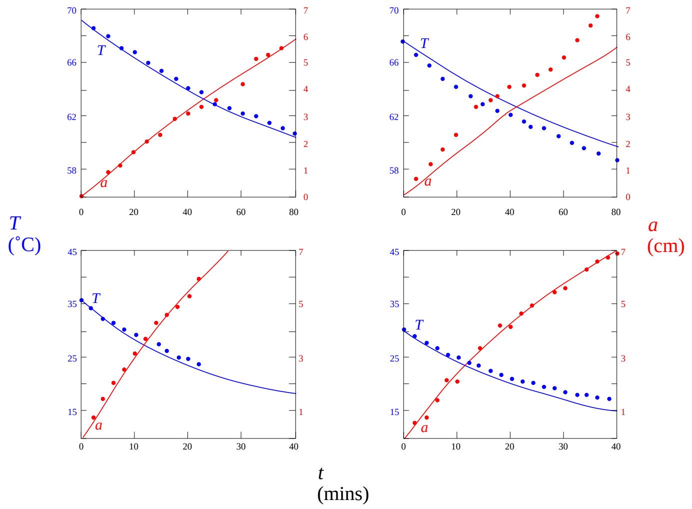
$$T = T_\infty + (T_m - T_\infty)\exp\left[-\frac{\dot{a}(z-a)}{\kappa}\right]$$

with $H = kT_z\big|_{z=s} + L_V\,\dot{a}$

$$\dot{a} = \frac{H}{[L_V + \rho c (T_m - T_\infty)]} = \frac{\text{heat input}}{\text{heat to raise temp and then melt}}$$

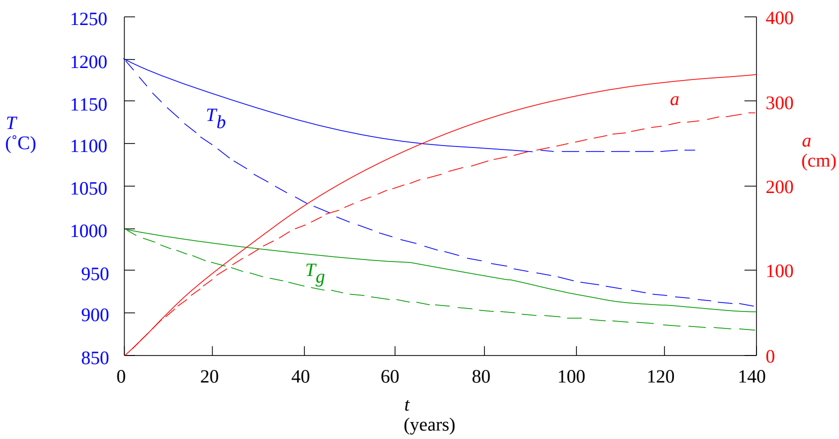
6. Experimental confirmation : heavy melt release 5.7

(H[2] and Sparks 1988)



Experimental and theoretical temperatures and melt thicknesses as functions of time: (a) and (b) wax roof; (c) and (d) ice roof experiments.

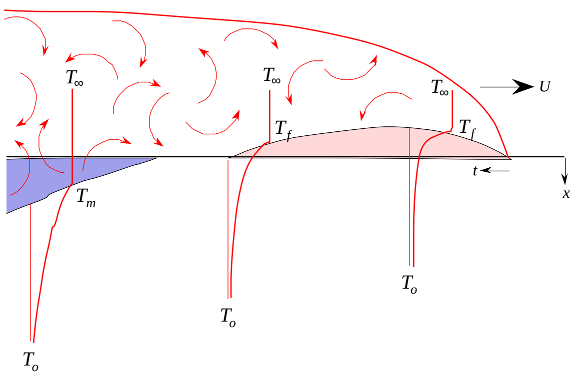
7. Geological application : light melt release 5.8



The mean temperatures in the basaltic (T_b) and granitic (T_g) layers and the thickness of the granite layer a as functions of time: solid line - perfect fluid case; dashed line - allowing for crystallization in both layers. The basalt cannot be considered a fluid below 1091 ˚C.

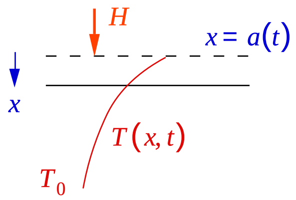
5.9

8. Consider the initial value problem of hot fluid suddenly introduced to flow turbulently over a cold solid ground. Does melting of solid or solidification of fluid occur first? Constant turbulent flux is to be compared with large (initially infinitely large) conductive flux, and so solidification must occur first (unless solidification temperature of fluid is less than initial temperature of ground).



5.10

Consider equal melting and freezing temperatures T_*



$$T_t = \kappa T_{xx} \qquad [x > a(t)]$$

$$T = T_* \qquad L_V \dot{a} = H + kT_x \qquad [x = a(t)]$$

$$T \to T_0 \qquad (x \to \infty)$$

$$T = T_0 \qquad (t = 0,\ x > 0)$$

Non-dimensionalise by introducing

$$T = T_0 + \left(T_* - T_0\right)\theta \qquad x = k\left(T_* - T_0\right)\frac{\xi}{H}$$

$$a = k\left(T_* - T_0\right)\frac{\eta}{H} \qquad t = \tfrac{1}{4}k^2\left(T_* - T_0\right)^2\frac{\tau}{\kappa H^2}$$

space scale

time scale

5.11

$$\theta_\tau = \tfrac{1}{4}\theta_{\xi\xi} \qquad [\xi > \eta(\tau)]$$

$$\theta = 1 \quad 4S\dot{\eta} = 1 + \theta_\xi \qquad [\xi = \eta(\tau)]$$

$$\theta \to 0 \qquad (\xi \to \infty)$$

$$\theta = 0 \qquad (\tau = 0, \xi > 0),$$

(H[2], 1989)

where the Stefan number $S = \dfrac{L}{c\left(T_* - T_0\right)}$,

the ratio of the latent heat of melting to the heat needed to raise the solid from its initial temperature to its melting temperature.

Nonlinear problem, with linearisation possible by writing $\xi = 0$ rather than $\xi = \eta$ for boundary condition

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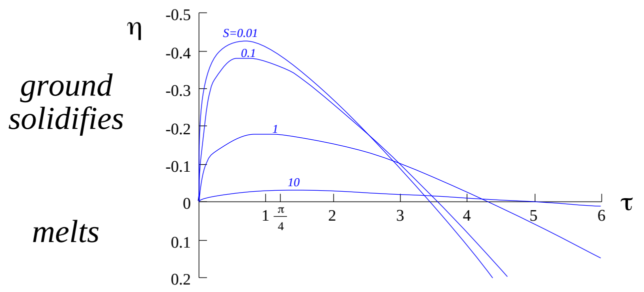
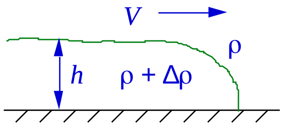
solution $\quad \theta(\xi,\tau) = \operatorname{erfc}\left(\xi/\tau^{\frac{1}{2}}\right) \qquad \eta(\tau) = \left[\tfrac{1}{4}\tau - \left(\dfrac{\tau}{\pi}\right)^{\frac{1}{2}}\right] S^{-1}$

i.e. η is initially negative, attains a minimum and thereafter increases steadily. The solution then very rapidly attains the steady state derived in 5.6.



9. Consider an inviscid gravity current of thickness h, 5.13

propagating at velocity V due to density difference $\Delta\rho$:



Important parameters

$V,\ h,\ \dfrac{\Delta\rho g}{\rho} = g'$ (reduced gravity)

$V \equiv LT^{-1}, \quad h \equiv L, \quad g' \equiv LT^{-2}$

There is one non-dimensional parameter, the Froude number

$$Fr = V/(g'h)^{\frac{1}{2}} = \frac{\text{velocity of current}}{\text{velocity of long waves on current}}$$

(c.f. Mach number)

For Re > c.500 $\quad \frac{1}{2}\rho V^2 = \Delta\rho g h$

i.e. $\quad Fr = \sqrt{2}$

In terms of $Q = Vh$

$$h = \tfrac{1}{2}\left(Q^2/g'\right)^{\frac{1}{3}}$$

(Note that low Reynolds number currents are different.)

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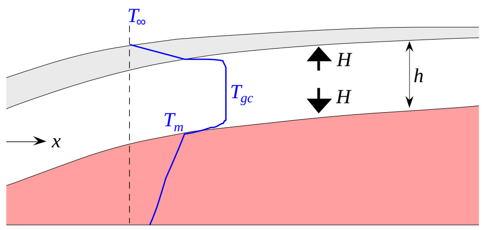
10. Consider now a hot turbulent gravity current flowing over an erodible bed and under a semi-infinite cool ocean (as occurred in the Archaean)

H^2 et al 1984; H^2 1986



A sketch of a hot, turbulent gravity current propagating over an erodible bed and forming a thin crust at the top of the current.

5.15

Assume for simplicity that: The melting temperature of the ground is identical to the freezing temperature of the lava; and that the far-field temperatures in the ground and the water are identical. Conservation of heat then states

$$\rho c_p h \dot{T}_{gc} = -2H - \rho c_p \dot{s}\left(T_{gc} - T_m\right)$$

rate of change of thermal capacity of current

heat lost from current

heat required to bring molten rock to mean temperature

In turbulent flows $H = h_T\left(T_{gc} - T_m\right)$

with h_T here assumed constant

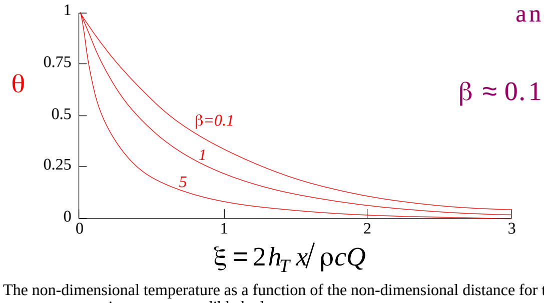
The solution is obtained by transforming t derivatives to x derivatives by $\frac{d}{dt} = V\frac{d}{dx}$ and introducing the non-dimensional quantities

$$\theta = \frac{T_{gc} - T_m}{T_{gc}(0) - T_m} \equiv \frac{T_{gc} - T_m}{\Delta} \qquad \xi = \frac{2h_T x}{\rho c Q}$$

5.16

$$\theta_\xi = -\theta - \beta\theta^2 \qquad \theta(0) = 1 \qquad \theta = \left[(\beta+1)e^\xi - \beta\right]$$

$$\beta = \frac{\Delta}{2\left(T_m - T_\infty + Lc^{-1}\right)} = \frac{\text{thermal energy of input}}{\text{thermal energy to raise ground to } T_m \text{ and melt it}}$$



$\beta \approx 0.1$ for geological situations

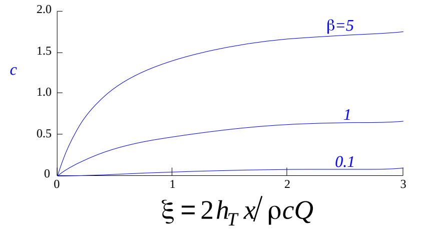
The non-dimensional temperature as a function of the non-dimensional distance for three values of b for a hot, turbulent gravity current propagating over an erodible bed,

The relative contamination

$$c = \int_0^x \dot{s}\, dx / Q$$

$$= \ln\left[(\beta+1)e^\xi - \beta\right] - \xi$$

$$\to \ln(\beta+1) \quad \text{as} \quad \xi \to \infty$$



The relative contamination as a function of the non-dimensional distance,

5.17

Main ideas of lecture 5

1 Preliminary equations

2 Heat transfer without melting

3-5 Melting

6-7 Melting the roof of a chamber

8 The initiation of melting

9 Inviscid gravity currents

10 Hot fluids over erodible cold surfaces : ancient lava flows

5.18

Lecture 5. Melting

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Hill, J.M., 1987 *One-dimensional Stefan Problems: An Introduction*, Longman.

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