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Lecture 4: Vortices – structure and propagation

- Gulf Stream and Kuroshio rings (and others)
- Cyclo-geostrophic balance
  Instabilities
- Beta-gyres
- Integral constraints
- Modons and hetons
  Vortex patches

1 —Gulf Stream and Kuroshio rings (and others)

Once eddies form, they propagate through the ocean, interacting with other eddies, mesoscale flows, topography, and, frequently, the jet itself. Agulhas rings can be tracked clear across the South Atlantic to Brazil. Some Gulf Stream cold core rings survive for several years. Kuroshio rings have been observed to merge (Yasuda, et al., 1992).

2 —Cyclo-geostrophic balance

$$\frac{1}{r}\frac{\partial}{\partial r}rv + f = \frac{H+\eta}{H}\big(f + q_0\mathcal{H}(a-r)\big)$$

or

$$\frac{1}{r}\frac{\partial}{\partial r}rv - f\frac{\eta}{H} = \frac{H+\eta}{H}q_0\mathcal{H}(a-r)$$

$$\frac{v^2}{r} + fv = gH\frac{\partial}{\partial r}\frac{\eta}{H}$$

Scale $v \sim \sqrt{HG}$, $r \sim \sqrt{gH}/f$, $\eta \sim H$, leaving $Ro = q_0/f$ and $\gamma = a/R_d$ as the parameters

$$\frac{1}{r}\frac{\partial}{\partial r}rv - \eta = (1+\eta)Ro\mathcal{H}(\gamma - r)$$

$$\frac{v^2}{r} + v = \frac{\partial}{\partial r}\eta$$

We can solve this easily with a shooting method starting with a guess of $\eta(0)$ and using $\frac{\partial}{\partial r}v \simeq \frac{1}{2}(\eta + Ro + \eta Ro)$ to start; $\eta(0)$ is adjusted to find a solution which is well-behaved at large r. The results can be compared to the QG version which drops the η on the r.h.s. of the PV equation and the cyclostrophic term. q=0.3 q=0.8 q=1.5 q=-0.3 q=-0.8 q=-1.5 q=-2.5

- for cyclones, the velocities and thickness perturbations are weaker than predicted by QG theory. If, however, we simply drop the centrifugal term, the equations become linear. The deformation radii inside and outside are different: $R_d^{\text{out}} = \sqrt{1 + Ro}\, R_d^{\text{in}}$. Note that the exact vorticity equation

$$\nabla^2 \psi = h' q + Hq - f_0$$

with the geostrophic approx. becomes

$$\nabla^2 \psi - \frac{f_0 q}{g} \psi = Hq - f_0$$

so that we are replacing a factor f_0/H by q in the definition of R_d. The velocities from this are very close, but the height field is a bit weaker since it does not have to compensate the centrifugal terms.

- for anticyclones, the velocities and thickness perturbations are stronger than QG; again with the variable R_d, the velocities are similar, but the heights are now weaker, since v^2/r is opposite in sign to fv. The Ro values are limited; when they become greater than 1, we have "anomalous highs" with a reversed central pressure.

We can try to invert more complex PV distributions, using an iterative technique. Start with

$$Q = Hq - f_0 \quad \Rightarrow \quad \nabla^2 \psi + f_0 + \beta y = f_0 \frac{h}{H} + \frac{h}{H} Q$$

and let $h = H + f_0 \psi / g + h'$; the PV equation becomes

$$\nabla^2 \psi - \gamma^2 \psi = Q - \beta y + \left[Q \frac{f_0 \psi}{gH} + Q \frac{h'}{H} + f_0 \frac{h'}{H} \right]$$

and the divergence/ balance equation yields

$$\nabla^2 g h' = \nabla \cdot (\nabla^2 \psi + \beta y) \nabla \psi - \frac{1}{2} \nabla^2 |\nabla \psi|^2 = \nabla \cdot \beta y \nabla \psi + 2[\psi_x, \psi_y]$$

We solve the PV equation for ψ, neglecting the terms in square brackets, then solve the balance eqn. for h'. We substitute these estimates into the square bracket terms and iterate. However, the procedure may not converge; for cyclones with Rossby number bigger than $\frac{1}{2}$ (at least for $\gamma = 1$) it diverges immediately. But it does converge with about 5 iterations for both cyclones and anticyclones when the Rossby number is less than about 0.2. a=-0.2 a=0.2 a=0.4 a=0.7 a=1.5 a=-0.4 a=-0.4 a=-0.7

3 —Instabilities

For linearized stability, we can write the usual Rayleigh equation; however, following the approach used for jets leads to

$$\frac{\partial}{\partial t}\eta + \imath m \left[\frac{\overline{v}(r)}{r}\eta(r) - \frac{1}{r}\int dr\, G_m(r|r')\overline{q}_r(r')\eta(r') \right] = 0$$

for modes with $\eta = \eta(r,t)\exp(\imath m\theta)$.

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \frac{m^2}{r^2} - \gamma^2 \right) G_m(r|r') = \delta(r - r') \tag{3.1}$$

Discretization leads to a matrix equation; for contour dynamics, it's

$$\frac{\partial}{\partial t}\eta_i + \frac{\partial}{\partial \theta} M_{ij}\eta_j = 0 \quad , \quad M_{ij} = \frac{\overline{v}_i}{r_i}\delta_{ij} + \frac{1}{r_i} G_m(r_i|r_j) q_j$$

Normal modes with

$$M_{ij} e_j = \omega\, e_i$$

behave like $\eta_i \propto \exp(-\imath\, m\, \omega t)$ so we can find growth rates and detailed stability conditions by examining the eigenvalues of M_{ij}.

Note that the basic state flow satisfies

$$\overline{v}_i = -G_1(r_i|r_j) q_j$$

with the Green's function given by *(G-EBT)* or *(G-BT)*.

We consider two contours, at $r_1 = 1$ and $r_2 = b$, and choose the PV jumps δ_1 and δ_2 such that $\overline{v}_1 = 1$ and $\overline{v}_2 = 0$. There are two parameters left: $\gamma = r_1/R_d$ and $b = r_2/r_1$. The vortex is unstable for b less than a critical value (2 in the $\gamma = 0$ case and decreasing for larger γ); the growth rates increase as b and γ decrease. b=2.1 b=1.9 bc vortex

4 —Beta-gyres

Linear Rossby waves are dispersive; even in the QG system, the PV equation

$$\frac{d}{dt}Q' - \beta\frac{\partial}{\partial x}[\gamma^2 - \nabla^2]^{-1}Q' = 0$$

shows that the speed will depend on the structure, although the dependence is weak for long waves. In that case, we can solve the inversion problem by iteration

$$\begin{aligned}\psi &= -R_d^2 Q + R_d^2\nabla^2\psi \\ &\simeq -R_d^2 Q - R_d^4\nabla^2 Q\end{aligned}$$

and the PV equation becomes

$$\frac{d}{dt}Q - \beta R_d^2\frac{\partial}{\partial x}(Q + R_d^2\nabla^2 Q) - R_d^2 J(Q, Q + R_d^2\nabla^2 Q) \simeq 0$$

We can switch to a frame of reference moving with the long wave speed

$$Q = Q(x - c_0 t, t)$$

with $c_0 = -\beta R_d^2$ and then find

$$\frac{d}{dt}Q - \beta R_d^4\frac{\partial}{\partial x}\nabla^2 Q - R_d^4 J(Q, \nabla^2 Q) = 0$$

Clearly, the linear term creates a structure which is different from Q or slightly shifted versions thereof. Once the radial symmetry is broken, the nonlinear terms can play a role.
bt linear bc linear bt nonlinear total pv bc nonlinear longer pv
-

To see how the planetary vorticity gradient affects an eddy, we can add the β-effect to the "vortex precession" model and examine the movement. We

$$Q = \mathcal{L}\psi + \beta y = Q_0\mathcal{H}(R + \xi - r) + \tilde{Q} + \beta y$$

and look in a moving frame

$$\psi^* = \psi + cy - c'x$$

with the translation speed to be determined. In the advection equation, the vortex part

$$\frac{D}{Dt}Q_0\mathcal{H}(R + \xi - r) = Q_0\delta(R + \xi - r)\frac{D}{Dt}\xi - u^* Q_0\delta(R + \xi - r)$$

has singularities which must be eliminated by making the interface a material surface

$$\frac{d}{dt}\xi + \frac{1}{r}\psi_r^*(R + \xi, \theta, t)\frac{\partial}{\partial\theta}\xi = -\frac{1}{r}\psi_\theta^*(R + \xi, \theta, t)$$

We further split the PV up into a circular vortex, the waves on the vortex, and the residual:

$$Q = Q_0 \mathcal{H}(R-r) + Q_0[\mathcal{H}(R+\xi-r) - \mathcal{H}(R-r)] + \tilde{Q} + \beta y$$

with associated streamfunctions

$$\mathcal{L}\overline{\psi} = Q_0\mathcal{H}(R-r) \quad , \quad \mathcal{L}\psi' = Q_0[\mathcal{H}(R+\xi-r) - \mathcal{H}(R-r)] \simeq Q_0 \xi \delta(R-r) \quad , \quad \mathcal{L}\tilde{\psi} = \tilde{Q}$$

Consistent with the linearization of the inversion relation for ψ', we linearize the kinematic condition

$$\frac{d}{dt}\xi = -\overline{\Omega}(R)\frac{\partial}{\partial\theta}\xi - \frac{1}{R}\frac{\partial}{\partial\theta}[\psi' + \tilde{\psi} + cR\sin\theta - c'R\cos\theta]$$

with $\overline{\Omega}(R) = \overline{v}(R)/R$. The residual equation has many terms

$$\frac{\partial}{\partial t}\tilde{Q} + \overline{\Omega}\frac{\partial}{\partial\theta}\tilde{Q} + \beta r\overline{\Omega}\cos\theta + J(\psi' + \tilde{\psi} + cy - c'x, \tilde{Q} + \beta y) = 0$$

The dominant terms in the vortex are

$$\frac{d}{dt}\tilde{Q} + \overline{\Omega}\frac{\partial}{\partial\theta}Q + \beta r\overline{\Omega}\cos\theta = 0$$

and these produce the "β-gyres"

$$\tilde{Q} = \beta r\left[\sin(\theta - \overline{\Omega}t) - \sin(\theta)\right]$$

$$\tilde{\psi} = \beta R_d^2 r\sin\theta - \beta\cos\theta\int r'dr'G_1(r|r')\sin(\overline{\Omega}(r)t) + \beta\sin\theta\int r'dr'G_1(r|r')\cos(\overline{\Omega}(r)t)$$

We can define the vortex position by requiring the $\sin\theta$ and $\cos\theta$ parts of ξ must vanish. But for these $\overline{\Omega}\xi_\theta + \frac{1}{R}\psi'_\theta = 0$, so that

$$\begin{aligned}
c &= -\beta R_d^2 - \beta\frac{1}{R}\int r'dr'G_1(R|r')\cos(\overline{\Omega}(r')t) \\
c' &= -\beta\frac{1}{R}\int r'dr'G_1(R|r')\sin(\overline{\Omega}(r')t) \\
c + \imath c' &= -\beta R_d^2 - \beta\frac{1}{R}\int r'dr'G_1(R|r')\exp(\imath\overline{\Omega}(r')t)
\end{aligned}$$

But

$$\overline{v}(r') = r'\overline{\Omega}(r') = -Q_0 G_1(r'|R) = -Q_0\frac{R}{r'}G_1(R|r')$$

so that

$$c + \imath c' = -\beta R_d^2 + \frac{\beta}{Q_0 R^2}\int dr'\, r'^3\overline{\Omega}(r')\exp(\imath\overline{\Omega}(r')t)$$

and

$$X + \imath Y = -\beta R_d^2 t + \imath \frac{\beta}{Q_0 R^2} \int dr'\, r'^3 \left[1 - \exp(\imath \overline{\Omega}(r')t)\right]$$

beta-gyre theory

NON-QG EFFECTS

We can get information on the propagation speeds for eddies with significant Rossby numbers directly from the shallow water equations. If we define the anomalous mass associated with the eddy

$$M = \int d\mathbf{x}\, \eta$$

with $\eta = h - H$ (positive for anticyclones, negative for cyclones), we can show from the mass conservation equation

$$\frac{\partial}{\partial t} M = 0$$

and

$$\frac{\partial}{\partial t}\mathbf{X} = \frac{1}{M} \int h\mathbf{u}$$

where $\mathbf{X}$ is the centroid of the mass anomaly

$$\mathbf{X} = \frac{1}{M} \int \mathbf{x}\eta$$

If we look at the centroid acceleration

$$\frac{\partial^2}{\partial t^2}\mathbf{X} = \frac{1}{M} \int \frac{\partial}{\partial t}(h\mathbf{u})$$

and use the transport equations

$$\frac{\partial}{\partial t}(hu_i) + \nabla_j \cdot (h\mathbf{u}_j u_i) + (\hat{\mathbf{z}} \times fh\mathbf{u})_i = -\nabla_i g\eta(H + \eta/2)$$

to show that

$$\frac{\partial^2}{\partial t^2}\mathbf{X} + \frac{1}{M}\hat{\mathbf{z}} \times \int fh\mathbf{u} = 0$$

For steadily propagating eddies, this reduces to the statement that the net Coriolis force over the eddy must vanish:

$$\int fh\mathbf{u} = 0$$

However, the mass equation gives

$$\nabla \cdot [h\mathbf{u} - \mathbf{c}\eta] = 0$$

so that

$$h\mathbf{u} = \mathbf{c}\eta + \hat{\mathbf{z}} \times \nabla\Psi$$

Thus we have

$$
\begin{aligned}
\mathbf{c}\int f\eta &= -\hat{\mathbf{z}}\times\int f\nabla\Psi \\
&= -\hat{\mathbf{z}}\times\int\nabla(f\Psi)+\hat{\mathbf{z}}\times\int\Psi\nabla f \\
&= \beta\hat{\mathbf{z}}\times\hat{\mathbf{y}}\int\Psi \\
&= -\hat{\mathbf{x}}\beta\int\Psi
\end{aligned}
$$

We therefore obtain the expression for the speed of steadily propagating eddies

$$
\mathbf{c} = -\hat{\mathbf{x}}\beta\frac{\int\Psi}{\int f\eta}
$$

If we use a geostrophic estimate for $\mathbf{u}$, we have

$$
\begin{aligned}
\mathbf{u}h &\simeq \hat{\mathbf{z}}\times\nabla\Psi \\
&\simeq \frac{1}{f_0}g\hat{\mathbf{z}}\times\nabla(H\eta+\frac{1}{2}\eta^2) \\
&= \hat{\mathbf{z}}\times\nabla\frac{gH}{f_0}(\eta+\frac{1}{2}\eta^2/H) \\
&\Rightarrow \\
\Psi &\simeq \frac{gH}{f_0}(\eta+\frac{1}{2}\eta^2/H)
\end{aligned}
$$

giving a speed estimate of

$$
\mathbf{c}\simeq-\beta\frac{gH}{f_0^2}\left(1+\frac{1}{2}\frac{\int(\eta/H)^2}{\int(\eta/H)}\right)\hat{\mathbf{x}}
$$

Thus anticyclones ($\eta > 0$) move faster than the linear long wave speed, while cyclones ($\eta < 0$) move slower.

Conservation of potential vorticity implies any steady motion must be east-west. But let us examine the equivalent of (4.xx) in the two-layer case.

The upper layer mass equation still gives

$$
\frac{\partial}{\partial t}M = 0 \quad , \quad \frac{\partial}{\partial t}\mathbf{X} = \frac{1}{M}\int h\mathbf{u}
$$

but the momentum equations now give

$$
\frac{\partial}{\partial t}hu_i + \nabla_j\cdot h\mathbf{u}_j u_i + (\hat{\mathbf{z}}\times fh\mathbf{u})_I = -\nabla_i g\eta(H+\eta/2) - (H+\eta)\nabla\phi_2
$$

or

$$\frac{\partial^2}{\partial t^2}\mathbf{X} + \frac{1}{M}\hat{\mathbf{z}} \times \int fh\mathbf{u} = -\frac{1}{M}(H+\eta)\nabla\phi_2$$

$$\frac{\partial^2}{\partial t^2}\mathbf{X} + f_0\hat{\mathbf{z}} \times \frac{\partial}{\partial t}\mathbf{X} + \frac{1}{M}\hat{\mathbf{z}} \times \int \beta yh\mathbf{u} = -\frac{1}{M}\int (H+\eta)\nabla\phi_2$$

implying that the centroid motion is a combination of inertial oscillations, beta-drift, and additional drift caused by the form drag of the lower layer. If the motion is quasi-steady

$$\frac{\partial}{\partial t}\mathbf{X} = -\frac{1}{f_0 M}\int \beta yh\mathbf{u} + \hat{\mathbf{z}} \times \frac{1}{f_0 M}\int (H+\eta)\nabla\phi_2$$

We can estimate the lower layer flows by using quasi-geostrophy since they are weak, and the interface displacements are small compared to the lower layer thickness:

$$\frac{D}{Dt}\zeta_2 + \beta y + f\eta/H_2 \simeq 0 \quad , \quad \phi_2 = f\psi_2$$

$$\frac{\partial}{\partial t}\mathbf{X} = -\frac{1}{f_0 M}\int \beta yh\mathbf{u} + \hat{\mathbf{z}} \times \frac{1}{M}\int (H+\eta)\nabla\psi_2$$

We can linearize this to

$$\frac{\partial}{\partial t}\nabla^2\psi_2 + \beta\frac{\partial}{\partial x}\psi_2 = -\frac{\partial}{\partial t}f\frac{\eta}{H_2}$$

For nearly westward propagation

$$\nabla^2\psi_2 + \frac{\beta}{|c|}\psi_2 = -f\frac{\eta}{H_2}$$

This is identical to the flow over topography problem; because of the radiation condition, the waves will not be symmetrical in the east-west direction. For an anticyclone ($\eta > 0$), we will generate a dipole moment which will push the vortex south ($\frac{\partial}{\partial x}\phi_2 < 0$). To see this, note that the radiation condition implies an outward flux of energy in the deep layer, which it must be getting from the upper layer.

$$\frac{\partial}{\partial t}\int |\nabla\psi_2|^2 - \oint \psi_2\hat{\mathbf{n}}\cdot\nabla\psi_2 - \frac{1}{2}\beta\oint \psi_2^2\hat{\mathbf{x}}\cdot\hat{\mathbf{n}} = \frac{f}{H}\int \psi_2\frac{\partial}{\partial t}\eta = -c\frac{f}{H}\int \psi_2\frac{\partial}{\partial x}\eta = \frac{cf}{H}\int \eta\frac{\partial}{\partial x}\psi_2$$

Energy input requires $c\int \eta\frac{\partial}{\partial x}\psi_2 > 0$; for $c < 0$, this is consistent with $\frac{\partial}{\partial t}Y < 0$.

5 —Integral constraints

When both layers are active the integral constraints have more implications. Summing the mass equations (with the rigid-lid assumption) gives

$$\nabla \cdot h_i \mathbf{u}_i = 0 \quad \Rightarrow \quad h_i \mathbf{u}_i = \hat{\mathbf{z}} \times \nabla \Psi$$

The northward momentum equations (in Cartesian coordinates)

$$\begin{aligned} \frac{\partial}{\partial t} h_1 v_1 + \nabla \cdot (h_1 v_1 \mathbf{u}_1) + f h_1 u_1 &= -\frac{\partial}{\partial y} \frac{1}{2} g' h_1^2 - h_1 \frac{\partial}{\partial y} \phi_2 \\ \frac{\partial}{\partial t} h_2 v_2 + \nabla \cdot (h_2 v_2 \mathbf{u}_2) + f h_2 u_2 &= -h_2 \frac{\partial}{\partial y} \phi_2 \end{aligned}$$

Adding these equations gives

$$\frac{\partial}{\partial t} \Psi_x + \nabla \cdot (h_1 v_1 \mathbf{u}_1 + h_2 v_2 \mathbf{u}_2) - f \Psi_y = -\frac{\partial}{\partial y} (\frac{1}{2} g' h_1^2 + H \phi_1)$$

and integrating, assuming all fields vanish rapidly enough, gives

$$\int d\mathbf{x}\, f \Psi_y = 0 \quad \Rightarrow \quad \beta \int d\mathbf{x}\, \Psi = 0$$

Applying similar arguments and assuming fields vanish sufficiently rapidly

$$\begin{aligned} \frac{\partial}{\partial t} h_1 v_1 + \nabla \cdot (h_1 v_1 \mathbf{u}_1) + f h_1 u_1 &= -\frac{\partial}{\partial y} \frac{1}{2} g' h_1^2 - h_1 \frac{\partial}{\partial y} \phi_2 \\ \frac{\partial}{\partial t} h_2 v_2 + \nabla \cdot (h_2 v_2 \mathbf{u}_2) + f h_2 u_2 &= -h_2 \frac{\partial}{\partial y} \phi_2 \end{aligned}$$

6 —Modons and hetons

The condition $\int \Psi = 0$ suggests that dipolar solutions may play a significant role in vortex dynamics. We shall look for steadily propagating solutions

$$\psi + cy = \Psi(Q + \beta y)$$

In the far field where Q and ψ decay, this requires

$$\Psi(Z) = \frac{c}{\beta} Z$$

so that

$$Q = \mathcal{L}(\psi) = \frac{\beta}{c} \psi \tag{6.1}$$

For $\mathcal{L} = \nabla^2 - \gamma^2$, we have

$$\nabla^2 \psi = \left(\gamma^2 + \frac{\beta}{c} \right) \psi = k^2 \psi$$

A decaying solution will be possible if $k^2 > 0$, implying $c > 0$ or $c < -\beta/\gamma^2$. The solutions are outside the linear wave range of speeds; this is to be expected for an imaginary wavenumber $c = \beta/(k^2 - \gamma^2)$.

The form (6.xx) must hold for all $Q + \beta y$ PV contours and corresponding $\psi + cy$ streamlines which extend to infinity; however it has no globally regular solutions. Instead, they are K Bessel functions. If there is a closed PV contour, we can use a different relationship inside and then match the two solutions (ψ and $\nabla\psi$) at the boundary.

The standard modon approach takes the closed streamline to be a circle and the PV-Ψ relationship to be a different linear function inside

$$Q + \beta y = -(\gamma^2 + K^2)(\psi + cy)$$

chosen to give oscillatory (J Bessel functions) inside. Note that γ^2 does not appear in the structure equations, but only int the expression for c; however, this is coupled into the structure because the separating streamline has

$$\psi + cr \sin\theta = \text{const.}$$

The simple solutions with a circular dividing streamline at $r = a$ are

$$\psi + cy = \begin{cases} \left(cr - ca \dfrac{K_1(kr)}{K_1(ka)} \right) \sin\theta & r > a \\ -c \dfrac{k^2}{K^2} \left(r - a \dfrac{J_1(Kr)}{J_1(Ka)} \right) \sin\theta & r < a \end{cases}$$

Matching the r-derivative gives an expression for K in terms of k; this leads to

$$\frac{K_2(ka)}{ka\, K_1(ka)} = -\frac{J_2(Ka)}{Ka\, J_1(Ka)}$$

therefore we have a family of solutions determined by the values of k, each of which has a translational velocity $c = \beta/(k^2 - \gamma^2)$.

There are similar two-layer baroclinic solutions, but they often have complicated structures, and, if they have a barotropic exterior component, move eastward (Flierl, et al., 1980). There are examples of 3D modons in the interior of a uniformly stratified fluid (19xx) and in a two layer model with vertical shear (Flierl, 19xx).

6.1 — Vortex patches

Another approach is more akin to contour dynamics: we start with the point vortex dipole on the beta-plane. We then "desingularize" by replacing the points with finite area, constant PV patches having the same integrated anomaly. As we do this, the original K_0 functions will begin to include K_n functions; the amplitudes are solved by requiring the patch boundaries to be to have constant $\psi + cy$ and continuity of tangential velocity.

While we could solve this problem analytically for small areas and numerically by root-finding or relaxation in the larger area case, we can also set the problem up as a Dirac-bracket, simulated annealing problem. We use Q fields which are close to step functions

$$Q_i = q_0 e^{-(r/r_0)^{10}}$$

We use

$$\mathcal{C}_1 = -\int y\, Q \quad , \quad \mathcal{C}_2 = \int xy\, Q$$

as the constraints. The patches quickly settle to a flattened oval shape; these solutions certainly indicate that patches with a discontinuity will exist.

References

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