ロスビー波 (2 次元非発散球面)

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1 支配方程式

1.1 球面 2 次元非発散方程式

支配方程式は次のように書きくだせる (Appendix 参照).

$$
\begin{aligned}
\frac{\partial}{\partial t}u + \frac{u}{a\cos\phi}\frac{\partial u}{\partial\lambda} + \frac{v}{a}\frac{\partial u}{\partial\phi} - \frac{\tan\phi uv}{a} - 2\Omega\sin\phi v &= -\frac{1}{\rho_0}\frac{1}{a\cos\phi}\frac{\partial}{\partial\lambda}p + f_\lambda, && (1)\\
\frac{\partial}{\partial t}v + \frac{u}{a\cos\phi}\frac{\partial v}{\partial\lambda} + \frac{v}{a}\frac{\partial v}{\partial\phi} + \frac{\tan\phi u^2}{a} + 2\Omega\sin\phi u &= -\frac{1}{\rho_0}\frac{1}{a}\frac{\partial}{\partial\phi}p + f_\phi, && (2)\\
\frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}u + \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\cos\phi v &= 0. && (3)
\end{aligned}
$$

ただし,

$(\lambda,\ \phi)$	(経度, 緯度),
$(u,\ v)$	速度 (東向き成分, 北向き成分)
Ω	系 (球殻) の自転角速度
a	球殻の半径
$(f_\lambda,\ f_\phi)$	外力, 粘性散逸項
p	圧力
ρ_0	密度 (定数)

適当な速度スケール U を導入することにより次のような無次元化をおこない, 世界を半径 1 の球面に規格化する:

速度スケール	U
空間スケール	a
時間スケール	$\frac{a}{U}$

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[0] 本編は/参照基礎/地球流体/線型波動/ロスビー波/ に位置するものである.

規格化された方程式系は次のようになる:

$$\begin{aligned}
\frac{\partial}{\partial t^*}u^* + \frac{u^*}{\cos\phi}\frac{\partial u^*}{\partial\lambda} + v^*\frac{\partial u^*}{\partial\phi} - \tan\phi u^* v^* - 2\Omega^*\sin\phi v^* &= -\frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}p^* + f^*_\lambda, \\
\frac{\partial}{\partial t^*}v^* + \frac{u^*}{\cos\phi}\frac{\partial v^*}{\partial\lambda} + v^*\frac{\partial v^*}{\partial\phi} + \tan\phi u^{*2} + 2\Omega^*\sin\phi u^* &= -\frac{\partial}{\partial\phi}p^* + f^*_\phi, \\
\frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}u^* + \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\cos\phi v^* &= 0.
\end{aligned}$$

ただし

$$\begin{aligned}
\Omega^* &\equiv \Omega\frac{a}{U} && \text{無次元化された系の自転角速度,} \\
p^* &\equiv \frac{p}{\rho_0 U^2} && \text{無次元化された圧力,} \\
f^*_{[\lambda,\phi]} &\equiv f_{[\lambda,\phi]}\frac{a}{U^2} && \text{無次元化された外力.}
\end{aligned}$$

$*$ を省略して書くことにすれば,

$$\boxed{\begin{aligned}
\frac{\partial}{\partial t}u + \left(u\frac{1}{\cos\phi}\frac{\partial}{\partial\lambda} + v\frac{\partial}{\partial\phi}\right)u - \tan\phi uv - 2\Omega\sin\phi v &= -\frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}p + f_\lambda, && (4) \\
\frac{\partial}{\partial t}v + \left(u\frac{1}{\cos\phi}\frac{\partial}{\partial\lambda} + v\frac{\partial}{\partial\phi}\right)v + \tan\phi u^2 + 2\Omega\sin\phi u &= -\frac{\partial}{\partial\phi}p + f_\phi, && (5) \\
\frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}u + \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\cos\phi v &= 0. && (6)
\end{aligned}}$$

粘性散逸項の表現例として, 通常の非発散歪みテンソルの表現を球面 2 次元化したものを用いれば

$$\begin{aligned}
f_\lambda &= \nu\left[(\nabla_h^2 + 2)u - \frac{2\sin\phi}{\cos^2\phi}\frac{\partial v}{\partial\lambda} - \frac{u}{\cos^2\phi}\right], && (7) \\
f_\phi &= \nu\left[(\nabla_h^2 + 2)v + \frac{2\sin\phi}{\cos^2\phi}\frac{\partial u}{\partial\lambda} - \frac{v}{\cos^2\phi}\right]. && (8)
\end{aligned}$$

詳細は Appendix を参照されたい.

1.2　流線関数と渦度方程式

(6) で記されるように非発散系を扱っているので 流線関数ψ を導入する:

$$\begin{aligned}
u &\equiv -\frac{\partial\psi}{\partial\phi}, && (9) \\
v &\equiv \frac{1}{\cos\phi}\frac{\partial\psi}{\partial\lambda}. && (10)
\end{aligned}$$

相対渦度 ζ, 絶対渦度 q はそれぞれ

$$\begin{aligned}
\zeta &\equiv \frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}v - \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi u) \\
&= \left[\frac{1}{\cos^2\phi}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\phi}\right)\right]\psi \\
&= \nabla_h^2\psi, \qquad (11) \\
q &\equiv \zeta + 2\Omega\sin\phi \qquad (12)
\end{aligned}$$

となる. ただし, ∇_h^2 は球面上のラプラシアン,

$$\nabla_h^2 \equiv \frac{1}{\cos^2\phi}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\phi}\right) \qquad (13)$$

である.

運動方程式 (4), (5) から渦度方程式を導き, 流線関数を用いて表現すれば,

$$\frac{\partial}{\partial t}\nabla_h^2\psi - \frac{1}{\cos\phi}\frac{\partial\psi}{\partial\phi}\frac{\partial\nabla_h^2\psi}{\partial\lambda} + \frac{1}{\cos\phi}\frac{\partial\psi}{\partial\lambda}\frac{\partial\nabla_h^2\psi}{\partial\phi} + 2\Omega\frac{\partial\psi}{\partial\lambda} = f_q. \qquad (14)$$

ここで, f_q は外力, 粘性による渦度生成消滅項で

$$f_q \equiv \frac{1}{\cos\phi}\frac{\partial f_\phi}{\partial\lambda} - \frac{1}{\cos\phi}\frac{\partial\cos\phi f_\lambda}{\partial\phi} \qquad (15)$$

である. 粘性散逸項の表現例として, 運動方程式での例に対応するものをあげておくと

$$f_q = \nu(\nabla_h^2 + 2)\nabla_h^2\psi. \qquad (16)$$

「+2」に注意 (Appendix を参照).

反対称作用素

$$J(X,Y) \equiv \frac{\partial X}{\partial\lambda}\frac{\partial Y}{\partial\phi} - \frac{\partial Y}{\partial\lambda}\frac{\partial X}{\partial\phi} \qquad (17)$$

を用いて記せば

$$\frac{\partial}{\partial t}\zeta + \frac{1}{\cos\phi}J(\psi,\zeta) + 2\Omega\frac{\partial\psi}{\partial\lambda} = f_q, \qquad (18)$$

あるいは,

$$\frac{\partial}{\partial t}q + \frac{1}{\cos\phi}J(\psi,q) = f_q. \qquad (19)$$

2　保存則

2.1　角運動量保存則

momentum

運動方程式の λ 成分は角運動量保存則に他ならない. 角運動量 $(u+\Omega\cos\phi)\cos\phi$ が陽に現れる形で書き換えると

$$\left[\frac{\partial}{\partial t}+u\frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}+v\frac{\partial}{\partial\phi}\right]\{(u+\Omega\cos\phi)\cos\phi\} \;=\; -\frac{\partial}{\partial\lambda}p+f_\lambda\cos\phi. \tag{20}$$

あるいはフラックス形で書いて

$$\begin{aligned}\frac{\partial}{\partial t}(u\cos\phi)+\frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}(u^2\cos\phi)+\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\{v(u+\Omega\cos\phi)\cos^2\phi\}&\\ = -\frac{\partial}{\partial\lambda}p+f_\lambda\cos\phi.&\end{aligned} \tag{21}$$

渦度擾乱 (ロスビー波) による角運動量のやりとりを考えるために, 渦度を用いた表現に変形すれば[1]

$$\frac{\partial}{\partial t}(u\cos\phi)+\frac{\partial}{\partial\lambda}\frac{u^2+v^2}{2}-(2\Omega\sin\phi+\zeta)v\cos\phi \;=\; -\frac{\partial}{\partial\lambda}p+f_\lambda\cos\phi. \tag{22}$$

東西平均 "$\overline{}$" を

$$\overline{} \;\equiv\; \frac{1}{2\pi}\int_0^{2\pi}d\lambda \tag{23}$$

で定義すれば, 東西平均角運動量 (いわゆる角運動量) の保存則は[2]

$$\boxed{\frac{\partial}{\partial t}(\overline{u}\cos\phi)-\overline{v\zeta}\cos\phi \;=\; \overline{f_\lambda}\cos\phi.} \tag{24}$$

コリオリと同じ [illegible]

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[1] 一般には

$$\frac{\partial\boldsymbol{v}}{\partial t}+(2\boldsymbol{\Omega}+\boldsymbol{\zeta})\times\boldsymbol{v}+\nabla\frac{\boldsymbol{v}^2}{2} \;=\; -\frac{1}{\rho}\nabla p+\boldsymbol{f}.$$

[2] このことは渦度方程式 (18) の東西平均を計算しても確かめられる. 実際

$$\begin{aligned}\overline{\zeta} &= -\frac{1}{\cos\phi}\frac{\partial\overline{u}\cos\phi}{\partial\phi},\\ \overline{\frac{1}{\cos\phi}J(\psi,\zeta)} &= \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left(\overline{\frac{\partial\psi}{\partial\lambda}\zeta}\right),\end{aligned}$$

角運動量流速の収束は

$$\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}(\overline{(u+\Omega\cos\phi)v}\cos^2\phi) \;=\; -\overline{vq}\cos\phi \;=\; -\overline{v\zeta}\cos\phi \qquad (25)$$

であることに注意.

2.2　運動エネルギー保存則

運動エネルギー $(u^2+v^2)/2$ の保存則は運動方程式から直ちに得られる:

$$\left[\frac{\partial}{\partial t}+u\frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}+v\frac{\partial}{\partial\phi}\right]\frac{u^2+v^2}{2} \;=\; -\frac{1}{\cos\phi}\frac{\partial up}{\partial\lambda}-\frac{1}{\cos\phi}\frac{\partial\cos\phi vp}{\partial\phi}+f_\lambda u+f_\phi v. \qquad (26)$$

あるいはフラックス形で書いて

$$\begin{aligned}\frac{\partial}{\partial t}\frac{u^2+v^2}{2}+\frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}\left[u\frac{u^2+v^2}{2}\right]+\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left[\cos\phi v\frac{u^2+v^2}{2}\right]&\\ = -\frac{1}{\cos\phi}\frac{\partial up}{\partial\lambda}-\frac{1}{\cos\phi}\frac{\partial\cos\phi vp}{\partial\phi}+f_\lambda u+f_\phi v.& \qquad (27)\end{aligned}$$

しかしながらこの形式はあまり用いられない. 通常は圧力 p を消去した形式が用いられる. 渦度方程式 (14) に ψ をかけて変形すれば

$$\begin{aligned}\frac{\partial}{\partial t}\frac{1}{2}\left[\left(\frac{1}{\cos\phi}\frac{\partial\psi}{\partial\lambda}\right)^2+\left(\frac{\partial\psi}{\partial\phi}\right)^2\right]+\frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}\left[q\frac{\partial}{\partial\phi}\frac{\psi^2}{2}+\psi\frac{1}{\cos\phi}\frac{\partial}{\partial t}\frac{\partial\psi}{\partial\lambda}+\psi f_\phi\right]&\\ -\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left[\cos\phi\left(q\frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}\frac{\psi^2}{2}-\psi\frac{\partial}{\partial t}\frac{\partial\psi}{\partial\phi}+\psi f_\lambda\right)\right]&\\ = -\frac{\partial\psi}{\partial\phi}f_\lambda+\frac{1}{\cos\phi}\frac{\partial\psi}{\partial\lambda}f_\phi& \qquad (28)\end{aligned}$$

$$\overline{f_q} \;=\; -\frac{1}{\cos\phi}\frac{\partial\overline{f_\lambda}\cos\phi}{\partial\phi}$$

であるから, (18) の東西平均は

$$-\frac{\partial}{\partial t}\frac{1}{\cos\phi}\frac{\partial\overline{u}\cos\phi}{\partial\phi}+\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left(\overline{\frac{\partial\psi}{\partial\lambda}\zeta}\right) \;=\; -\frac{1}{\cos\phi}\frac{\partial\overline{f_\lambda}\cos\phi}{\partial\phi}.$$

極での境界条件を使って積分すれば

$$-\frac{\partial}{\partial t}(\overline{u}\cos\phi)+\overline{\frac{\partial\psi}{\partial\lambda}\zeta} \;=\; -\overline{f_\lambda}\cos\phi$$

これは (24) にほかならない.

2.3　エンストロフィー保存則

Casimir

絶対エンストロフィー $q^2/2$ の保存則は渦度方程式 (19) に q をかけることにより直ちに得られる:

$$\frac{\partial}{\partial t}\frac{q^2}{2} + u\frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}\frac{q^2}{2} + v\frac{\partial}{\partial\phi}\frac{q^2}{2} \;=\; qf_q, \tag{29}$$

あるいは

$$\frac{\partial}{\partial t}\frac{q^2}{2} - \frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}\left[\frac{q^2}{2}\frac{\partial\psi}{\partial\phi}\right] + \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left[\frac{q^2}{2}\frac{\partial\psi}{\partial\lambda}\right] \;=\; qf_q. \tag{30}$$

相対渦度に対するエンストロフィー $\zeta^2/2$ で書き直すと

$$\frac{\partial}{\partial t}\frac{\zeta^2}{2} + u\frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}\frac{\zeta^2}{2} + v\frac{\partial}{\partial\phi}\frac{\zeta^2}{2} + 2\Omega\cos\phi v\zeta \;=\; \zeta f_q. \tag{31}$$

あるいは

$$\frac{\partial}{\partial t}\frac{\zeta^2}{2} + \frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}\left[u\frac{\zeta^2}{2} - 2\Omega\cos\phi\frac{u^2-v^2}{2}\right] + \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left[\cos\phi v\frac{\zeta^2}{2} - 2\Omega\cos^2\phi uv\right] \;=\; \zeta f_q. \tag{32}$$

すなわち

$$\frac{\partial}{\partial t}\frac{\zeta^2}{2} - \frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}\left[\frac{\zeta^2}{2}\frac{\partial\psi}{\partial\phi} + \Omega\cos\phi\left\{\left(\frac{\partial\psi}{\partial\phi}\right)^2 - \left(\frac{1}{\cos\phi}\frac{\partial\psi}{\partial\lambda}\right)^2\right\}\right] + \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left[\frac{\zeta^2}{2}\frac{\partial\psi}{\partial\lambda} + 2\Omega\cos\phi\frac{\partial\psi}{\partial\lambda}\frac{\partial\psi}{\partial\phi}\right] \;=\; \zeta f_q. \tag{33}$$

2.4　その他の有用な保存則

Momentum Casimir

角運動量保存則 (24) とエンストロフィー保存則 (31) を東西平均したものとを組み合わせ, $\overline{v\zeta}$ を消去すると

$$\frac{\partial}{\partial t}\left[\overline{u}\cos\phi + \frac{\overline{\zeta^2}}{4\Omega}\right] + \frac{1}{\cos\phi}\overline{J(\psi, \frac{\zeta^2}{4\Omega})} \;=\; \overline{\frac{\zeta}{2\Omega}f_q} + \overline{f_\lambda}\cos\phi. \tag{34}$$

これはいわゆる擬角運動量の一種である (にちがいない).

3　線型, 弱非線形理論

林

3.1　展開

世界を軸対象な基本流 $\overline{u} = \overline{u}(\phi)$ と擾乱とにわけ, 擾乱を振幅展開する.

U

$$
\begin{aligned}
u &= \overline{u} + u' + u^{(2)} + \cdots, &(35)\\
v &= \phantom{\overline{u} +{}} v' + v^{(2)} + \cdots, &(36)\\
\zeta &= \overline{\zeta} + \zeta' + \zeta^{(2)} + \cdots, &(37)\\
\psi &= \overline{\psi} + \psi' + \psi^{(2)} + \cdots. &(38)
\end{aligned}
$$

ただし,

$$
\begin{aligned}
\overline{u} &= -\frac{\partial \overline{\psi}}{\partial \phi}, &(39)\\
u' &= -\frac{\partial \psi'}{\partial \phi}, \quad v' = \frac{1}{\cos\phi}\frac{\partial \psi'}{\partial \lambda}, &(40)\\
u^{(2)} &= -\frac{\partial \psi^{(2)}}{\partial \phi}, \quad v^{(2)} = \frac{1}{\cos\phi}\frac{\partial \psi^{(2)}}{\partial \lambda}, &(41)\\
&\cdots,
\end{aligned}
$$

また,

$$
\begin{aligned}
\overline{\zeta} &= -\frac{1}{\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\,\overline{u}) = \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial\overline{\psi}}{\partial\phi}\right), &(42)\\
\zeta' &= \frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}v' - \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\,u') = \nabla_h^2\psi', &(43)\\
\zeta^{(2)} &= \frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}v^{(2)} - \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi\,u^{(2)}) = \nabla_h^2\psi^{(2)}, &(44)\\
&\cdots
\end{aligned}
$$

である.

3.2 渦度方程式の振幅展開

- 渦度方程式(18)に，振巾展開された表現を代入すると．

$$\boxed{\frac{\partial}{\partial t}\zeta' + \bar{u}\frac{1}{\cos\varphi}\partial_\lambda\zeta' + \frac{1}{\cos\varphi}\frac{\partial}{\partial\lambda}\psi'\,\hat{\beta} = f'_q}$$

$$\begin{aligned}\hat{\beta} &\equiv \frac{\partial}{\partial\varphi}(2\Omega\sin\varphi + \bar{\zeta}) \\ &= \frac{\partial}{\partial\varphi}\left[2\Omega\sin\varphi - \frac{1}{\cos\varphi}\partial_\varphi(\cos\varphi\,\bar{u})\right] \\ &= 2\Omega\cos\varphi - \frac{\partial}{\partial\varphi}\left[\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}(\cos\varphi\,\bar{u})\right] \end{aligned} \qquad (45)$$

$$\frac{\partial}{\partial t}\zeta^{(2)} + \bar{u}\frac{1}{\cos\varphi}\partial_\lambda\zeta^{(2)} + \frac{1}{\cos\varphi}\partial_\varphi\psi^{(2)}\,\hat{\beta} + \frac{1}{\cos\varphi}J(\psi',\zeta') = f_q^{(2)} \qquad (46)$$

- 軸対称成分に関しては，"‾"をとって，

$$\frac{\partial}{\partial t}\overline{\zeta'} = \overline{f'_q} \qquad (47)$$

$$\frac{\partial}{\partial t}\overline{\zeta^{(2)}} + \frac{1}{\cos\varphi}\overline{J(\psi',\zeta')} = \overline{f_q^{(2)}} \qquad (48)$$

軸対称成分に関しては, 振巾の1次のオーダーでの擾乱は、初期値と強制項 $\overline{f_\zeta}$ の構造のみによって完全にきまる.

振幅の2次のオーダーを, 角運動量でかきかえれば:

$$\frac{\partial}{\partial t}\frac{\partial}{\partial \varphi}(\cos\varphi\,\overline{u^{(2)}}) - \frac{\partial}{\partial \varphi}\left(\overline{\frac{\partial \psi'}{\partial \lambda}\zeta'}\right) = -\cos\varphi\,\overline{f_\zeta^{(1)}} \qquad (49)$$

φ で積分すれば

$$\frac{\partial}{\partial t}\cos\varphi\,\overline{u^{(2)}} - \overline{\frac{\partial \psi'}{\partial \lambda}\zeta'} = \cos\varphi\,\overline{f_\lambda^{(2)}} \qquad (50)$$

$$\left(\text{ただし}\quad f_\zeta^{(2)} = \frac{1}{\cos\varphi}\partial_\lambda f_\varphi^{(2)} - \frac{1}{\cos\varphi}\partial_\varphi \cos\varphi f_\lambda^{(2)}\right)$$

i.e.,

$$\boxed{\frac{\partial}{\partial t}\cos\varphi\,\overline{u^{(2)}} - \cos\varphi\,\overline{v'\zeta'} = \cos\varphi\,\overline{f_\lambda^{(2)}} \qquad (51)}$$

この結果は, 角運動量保存則 (24) からただちに得られる結果である

$\cos\varphi\,\overline{u^{(2)}}$ は 1次の擾乱を作るために必要な角運動量である, と解釈される.

3.3. 保存則（2次の量に関する）

渦度保存則（1次）を変形し，保存量を導く

ζ'をかけて

$$\partial_t \frac{\zeta'^2}{2} + \bar{u}\frac{1}{\cos\varphi}\partial_\lambda \frac{\zeta'^2}{2} + \frac{1}{\cos\varphi}\partial_\lambda\psi' \cdot \zeta'\hat{\beta} = \zeta' f'_g \qquad (52)$$

左辺第三項は

$$\begin{aligned}\partial_\lambda\psi'\zeta' &= \partial_\lambda\psi' \cdot \nabla_h\cdot(\nabla_h\psi') \\ &= \nabla_h\cdot(\partial_\lambda\psi'\nabla_h\psi') - \partial_\lambda\nabla_h\psi'\cdot\nabla_h\psi' \\ &= \nabla_h\cdot(\partial_\lambda\psi'\nabla_h\psi') - \frac{1}{2}\partial_\lambda(\nabla_h\psi')^2 \qquad (53)\end{aligned}$$

であるから．

$$\partial_t\frac{\zeta'^2}{2} + \frac{1}{\cos\varphi}\partial_\lambda\bar{u}\frac{\zeta'^2}{2} + \left[\nabla_h\cdot(\partial_\lambda\psi'\nabla_h\psi') - \frac{1}{2}\partial_\lambda(\nabla_h\psi')^2\right]\frac{\hat{\beta}}{\cos\varphi} = \zeta' f'_g \qquad (54)$$

$\partial_t\hat{\beta}$ は2次の量として無視すれば，

$$\partial_t\left[\frac{\zeta'^2}{2}\frac{\cos\varphi}{\hat{\beta}}\right] + \frac{1}{\cos\varphi}\partial_\lambda\left[\bar{u}\frac{\zeta'^2}{2}\frac{\cos\varphi}{\hat{\beta}} + \frac{1}{2}\cos\varphi\left[\left(\frac{1}{\cos\varphi}\frac{\partial\psi'}{\partial\lambda}\right)^2 - \left(\frac{\partial\psi'}{\partial\varphi}\right)^2\right]\right] + \frac{1}{\cos\varphi}\partial_\varphi\left[\cos\varphi\frac{\partial\psi'}{\partial\lambda}\frac{\partial\psi'}{\partial\varphi}\right] = \frac{\cos\varphi}{\hat{\beta}}\zeta' f'_g \qquad (55)$$

従って.

$$A \equiv \frac{\zeta'^2}{2}\frac{\cos\varphi}{\hat{\beta}} \qquad (56)$$

time independent

$$\mathbb{F} \equiv \left[A\bar{u} + \frac{1}{2}\cos\varphi\left[\frac{1}{\cos\varphi}\left(\frac{\partial\psi'}{\partial\lambda}\right)^2 - \left(\frac{\partial\psi'}{\partial\varphi}\right)^2\right], \frac{\partial\psi'}{\partial\lambda}\frac{\partial\psi'}{\partial\varphi}\right]$$

$$= \left(A\bar{u} + \frac{1}{2}\cos\varphi\left(v'^2 - u'^2\right), \cos\varphi\, u'v'\right] \qquad (57)$$

を定ぎすれば

$$\frac{\partial A}{\partial t} + \nabla_h\cdot\mathbb{F} = \frac{\cos\varphi}{\hat{\beta}}\zeta' f'_\zeta \qquad (58)$$

となる.

(52)の

・東西平均をとって.

$$\partial_t \overline{\frac{\zeta'^2}{2}}\frac{\cos\varphi}{\hat{\beta}} + \cos\varphi\,\overline{v'\zeta'} = \frac{\hat{\beta}}{\cos\varphi}\overline{\zeta' f'_\zeta} \qquad (59)$$

角運動量の2次の式(51)とくみあわせ, $\overline{v'\zeta'}$ を消去すれば.

$$\boxed{\frac{\partial}{\partial t}\left[\cos\varphi\,\overline{u^{(2)}} + \overline{\frac{\zeta'^2}{2}}\frac{\cos\varphi}{\hat{\beta}}\right] = \cos\varphi\,\overline{f_\lambda^{(2)}} + \frac{\cos\varphi}{\hat{\beta}}\overline{\zeta' f'_\zeta}} \qquad (60)$$

pseudo momentum

これをもって

$$-\overline{A} \equiv \frac{\overline{\xi'^2}}{2}\cdot\frac{\cos\varphi}{\beta} \qquad (61)$$

を擬角運動量という.

$\int -\overline{A}\,dS$ は，擾乱 ξ' を生成せしめるために系にかえねばならない角運動量に他ならない.

なお，この表現 (60) と (34) とでは似て非なることに注意されたい. (34) では

$$\frac{\xi'^2}{2}\frac{1}{2\Omega} \qquad (62)$$

が角運動量に代わるべき量であった.

一方

$$-\overline{A} = \frac{\overline{\xi'^2}}{2}\frac{\cos\varphi}{2\Omega\cos\varphi - \frac{\partial}{\partial\varphi}\left[\frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}(\cos\varphi\,\overline{u})\right]} \qquad (63)$$

両者は $\overline{u}=0$ のときにのみ一致する.

この差は“基本場”のちがいによるものである.
(34) では いわば 静止状態 ($u=0$) が“基本場”なのである.

・保存量（振巾の2次の）をより一般的に導くには・・・

運動量－カシミール の方法　（エネルギー－カシミールの方法）を用いる．

Haynes 1988 JAS

4. WKB近似.

線形擾乱のふるまいを記述する. $(\psi_g' = 0$ で $(38))$

4.1 位相函数

位相関数 Θ と 小さいパラメタ $\varepsilon \ll 1$ を導入し,

世界を次のように記述する.

$$\psi = \sum_{n=0} \psi_n(\varphi, \lambda, t)\, e^{i\frac{\Theta(\varphi,\lambda,t)}{\varepsilon}} \qquad (70)$$

4.2. 波数, 振動数

波数, 振動数は 次のように 定義される.

$$\omega \equiv -\frac{1}{\varepsilon}\frac{\partial\Theta}{\partial t} \qquad (71)$$

$$k \equiv \frac{1}{\varepsilon}\frac{1}{\cos\varphi}\frac{\partial\Theta}{\partial\lambda}$$

$$l \equiv \frac{1}{\varepsilon}\frac{\partial\Theta}{\partial\varphi}.$$

4.3 分散関係

渦度方程式 (45) に代入して 整理する.

$$\frac{1}{\cos\varphi}\partial_\lambda \psi = \left[\frac{i}{\varepsilon}\frac{1}{\cos\varphi}\frac{\partial\Theta}{\partial\lambda}\psi_0 + \frac{1}{\cos\varphi}\frac{\partial\psi_0}{\partial\lambda} + \cdots\right] e^{i\frac{\Theta}{\varepsilon}}$$

$$\partial_\varphi \psi = \left[\frac{i}{\varepsilon}\frac{\partial\Theta}{\partial\varphi}\psi_0 + \frac{\partial\psi_0}{\partial\varphi} + \cdots\right] e^{i\frac{\Theta}{\varepsilon}}$$

$$\frac{1}{\cos^2\varphi}\partial_{\lambda\lambda}\psi = \left[-\frac{1}{\varepsilon^2}\frac{1}{\cos^2\varphi}\left(\frac{\partial\Theta}{\partial\lambda}\right)^2\psi_0 + \frac{i}{\varepsilon}\left(\frac{1}{\cos^2\varphi}\frac{\partial^2\Theta}{\partial\lambda^2}\psi_0 + \frac{2}{\cos^2\varphi}\frac{\partial\Theta}{\partial\lambda}\frac{\partial\psi_0}{\partial\lambda}\right), + \cdots\right] e^{i\frac{\Theta}{\varepsilon}}.$$

$$\frac{1}{\cos\varphi}\partial_\varphi(\cos\varphi\,\partial_\varphi\psi) = \left[-\frac{1}{\varepsilon^2}\left(\frac{\partial\Theta}{\partial\varphi}\right)^2\psi_0 + \frac{i}{\varepsilon}\left(\frac{1}{\cos\varphi}\left(\partial_\varphi \cos\varphi\,\partial_{\varphi\varphi}\Theta.\right)\psi_0 + 2\frac{\partial\Theta}{\partial\varphi}\frac{\partial\psi_0}{\partial\varphi}\right) + \cdots\right] e^{i\frac{\Theta}{\varepsilon}}$$

$O(1^3)$ のオーダーから.

$$i\omega(k^2+l^2) - \bar{u}\ ik(k^2+l^2) + i\ k\hat{\beta} = 0.$$

i.e.

$$\omega = \bar{u}k - \frac{\hat{\beta}k}{k^2+l^2} \qquad (72)$$

(72) はいわゆる β 面のロスビー波の分散関係である.

4.4 波said方程式

波数保存則は

$$\frac{\partial k}{\partial t} = \frac{\partial}{\partial t}\frac{1}{\cos\varphi}\frac{1}{\varepsilon}\frac{\partial\Theta}{\partial\lambda} = -\frac{1}{\cos\varphi}\frac{\partial\omega}{\partial\lambda}$$

$$\frac{\partial l}{\partial t} = \frac{\partial}{\partial t}\frac{1}{\varepsilon}\frac{\partial\Theta}{\partial\varphi} = -\frac{\partial\omega}{\partial\varphi}$$

$$\frac{1}{\cos\varphi}\frac{\partial l}{\partial\lambda} = \frac{1}{\cos\varphi}\frac{1}{\varepsilon}\frac{\partial\Theta}{\partial\lambda\partial\varphi} = \frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}(\cos\varphi k)$$

よって，$\omega = \omega\left(\frac{\partial\Theta}{\varepsilon\partial\lambda}, \frac{\partial\Theta}{\varepsilon\partial\varphi}, \varphi, \lambda\right)$ とみなして.

$$\frac{\partial}{\partial t}k = -\frac{1}{\cos\varphi}\frac{\partial\omega}{\partial\lambda}$$

$$= -\frac{1}{\cos\varphi}\left[\frac{\partial\omega}{\partial\frac{1}{\varepsilon}\Theta_\lambda}\frac{1}{\varepsilon}\frac{\partial^2\Theta}{\partial\lambda^2} + \frac{\partial\omega}{\partial\frac{1}{\varepsilon}\Theta_\varphi}\frac{1}{\varepsilon}\frac{\partial^2\Theta}{\partial\varphi\partial\lambda} + \frac{\partial\omega}{\partial\lambda}\right]$$

$$= -\frac{1}{\cos\varphi}\left[\frac{\partial\omega}{\partial(k\cos\varphi)}\frac{\partial}{\partial\lambda}(k\cos\varphi) + \frac{\partial\omega}{\partial l}\frac{\partial}{\partial\varphi}(k\cos\varphi) + \frac{\partial\omega}{\partial\lambda}\right]$$

$$\frac{\partial}{\partial t}l = -\frac{\partial\omega}{\partial\varphi}$$

$$= -\left[\frac{\partial\omega}{\partial\frac{1}{\varepsilon}\Theta_\lambda}\frac{1}{\varepsilon}\frac{\partial^2\Theta}{\partial\varphi\partial\lambda} + \frac{\partial\omega}{\partial\frac{1}{\varepsilon}\Theta_\varphi}\frac{1}{\varepsilon}\frac{\partial^2\Theta}{\partial\varphi^2} + \frac{\partial\omega}{\partial\varphi}\right]$$

$$= -\left[\frac{\partial\omega}{\partial(k\cos\varphi)}\frac{\partial}{\partial\lambda}l + \frac{\partial\omega}{\partial l}\frac{\partial}{\partial\varphi}l + \frac{\partial\omega}{\partial\varphi}\right]$$

従って，群速度

$$C_{g\lambda} \equiv \frac{\partial\omega}{\partial(k\cos\varphi)}\cdot\cos\varphi$$

$$C_{g\varphi} \equiv \frac{\partial\omega}{\partial l}$$

を定ギすれば

$$\frac{\partial}{\partial t}(k\cos\varphi) + C_{g\lambda}\frac{1}{\cos\varphi}\frac{\partial}{\partial\lambda}(k\cos\varphi) + C_{g\varphi}\frac{\partial}{\partial\varphi}(k\cos\varphi) = -\frac{\partial\omega}{\partial\lambda}$$

$$\frac{\partial}{\partial t}l + C_{g\lambda}\frac{1}{\cos\varphi}\frac{\partial}{\partial\lambda}l + C_{g\varphi}\frac{\partial}{\partial\varphi}l = -\frac{\partial\omega}{\partial\varphi}$$

$$\frac{\partial}{\partial t}\omega + C_{g\lambda}\frac{1}{\cos\varphi}\frac{\partial}{\partial\lambda}\omega + C_{g\varphi}\frac{\partial}{\partial\varphi}\omega = \frac{\partial\omega}{\partial t}$$

となる.

群速度の具体的表現

$$c_{g\varphi} = \bar{u} + \frac{\hat{\beta}(k^2-l^2)}{(k^2+l^2)^2} = \bar{u} + (k^2-l^2)\frac{\hat{\omega}^2}{\hat{\beta}k^2}$$

$$c_{g\theta} = \frac{2\hat{\beta}kl}{(k^2+l^2)^2} = \frac{2l}{k}\frac{\hat{\omega}^2}{\hat{\beta}}$$

・主な特徴.

・基本場は φ のみの関数 なので，波線にそって

$$\left(\frac{\partial}{\partial t} + c_g\cdot\nabla\right)(k\cos\varphi) = 0$$

$$\left(\frac{\partial}{\partial t} + c_g\cdot\nabla\right) l = -\frac{\partial\omega}{\partial\varphi}$$

$$\left(\frac{\partial}{\partial t} + c_g\cdot\nabla\right)\omega = 0.$$

・いわゆる"波"の存在緯度は，ω, k に対して $l^2>0$ である条件として与えられる. i.e.,

$$l^2 = -\frac{\hat{\beta}k}{\omega-\bar{u}k} - k^2 > 0.$$

$k>0$ としておけば

$$-\frac{\hat{\beta}}{\omega-\bar{u}k} > k$$

波線にそって $k\cos\varphi = k_M$ (const) であることを使って

$$-\frac{2\Omega_M\cos^2\varphi}{\omega-\frac{\bar{u}}{\cos\varphi}k_M} > k_M \quad , \quad \Omega_M \equiv \frac{\hat{\beta}}{\cos\varphi} = \frac{1}{\cos\varphi}\partial_\varphi\bar{\zeta} + 2\Omega$$

$\varphi \to \pm \frac{\pi}{2}$ においては

$$\Omega_M < \infty , \quad \frac{\overline{u}}{\cos\varphi} < \infty$$

なので、不等式の|左辺| → 0. したがって、いわゆる転移緯度がどこかに存在する。

4.5 2次の量の伝播.

WKB すれば：

$$\frac{1}{2}\psi'^2\cos\varphi = \frac{1}{2}\frac{1}{(k^2+l^2)^2}\zeta'^2\cos\varphi$$

$$= \frac{A\hat{\beta}}{(k^2+l^2)^2}.$$

に注意して

$$\frac{\partial}{\partial t}A + \frac{1}{\cos\varphi}\frac{\partial}{\partial\varphi}\left[\left(\bar{u}+\frac{k^2-l^2}{(k^2+l^2)^2}\hat{\beta}\right)A\right]$$

$$+\frac{1}{\cos\varphi}\frac{\partial}{\partial\lambda}\left[\cos\varphi\cdot\frac{2kl}{(k^2+l^2)^2}\hat{\beta}A\right] = f'\zeta'\frac{\cos\varphi}{\hat{\beta}}$$

i.e.,

$$\frac{\partial}{\partial t}A + \nabla\cdot(C_g A) = f'\zeta'\frac{\cos\varphi}{\hat{\beta}}$$

平均すれば：

$$\frac{\partial}{\partial t}\bar{A} + \frac{1}{\cos\varphi}\partial_\varphi\cdot\left[\cos\varphi\, C_{g\varphi}\bar{A}\right] = \overline{f'\zeta'}\,\frac{\cos\varphi}{\hat{\beta}}.$$

A　球面座標

A.1　座標系と単位ベクトル

座標と対応する単位ベクトルを次のようにとることにする (図 A.1 参照[1]).

λ	e_λ	経度	$(0 \sim 2\pi)$
ϕ	e_ϕ	緯度	$(-\pi/2 \sim \pi/2)$
r	e_r	動系	

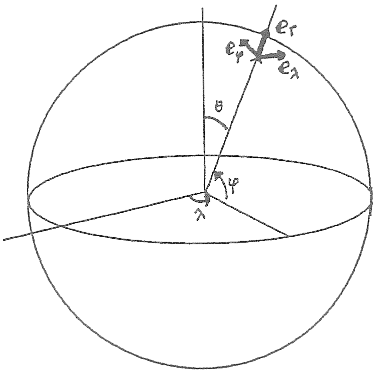

図 A.1 緯度経度球座標系

3 次元ユークリッド空間にうめこまれた状況なので

$$\begin{pmatrix} e_\lambda \\ e_\phi \\ e_r \end{pmatrix} = \begin{pmatrix} -\sin\lambda & \cos\lambda & 0 \\ -\sin\phi\cos\lambda & -\sin\phi\sin\lambda & \cos\phi \\ \cos\phi\cos\lambda & \cos\phi\sin\lambda & \sin\phi \end{pmatrix} \begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix}, \tag{45}$$

あるいは

$$\begin{pmatrix} e_x \\ e_y \\ e_z \end{pmatrix} = \begin{pmatrix} -\sin\lambda & -\sin\phi\cos\lambda & \cos\phi\cos\lambda \\ \cos\lambda & -\sin\phi\sin\lambda & \cos\phi\sin\lambda \\ 0 & \cos\phi & \sin\phi \end{pmatrix} \begin{pmatrix} e_\lambda \\ e_\phi \\ e_r \end{pmatrix}. \tag{46}$$

---

[1] 注意. 余緯度 $\theta \equiv \pi/2 - \phi$ と緯度ϕとの関係は次の通り.

$$\begin{aligned} \sin\theta &= \cos\phi \\ \frac{\partial}{\partial\theta} &= -\frac{\partial}{\partial\phi} \\ A_\theta &= -A_\phi \\ e_\theta &= -e_\phi \end{aligned}$$

ただし, A_θ, A_ϕ はベクトルの成分である.

A.2　単位ベクトルの微分

$$
\begin{aligned}
\frac{\partial}{\partial \lambda}\begin{pmatrix} \boldsymbol{e}_\lambda \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_r \end{pmatrix} &= \begin{pmatrix} \sin\phi \boldsymbol{e}_\phi - \cos\phi \boldsymbol{e}_r \\ -\sin\phi \boldsymbol{e}_\lambda \\ \cos\phi \boldsymbol{e}_\lambda \end{pmatrix}, & (47) \\
\frac{\partial}{\partial \phi}\begin{pmatrix} \boldsymbol{e}_\lambda \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_r \end{pmatrix} &= \begin{pmatrix} 0 \\ -\boldsymbol{e}_r \\ \boldsymbol{e}_\phi \end{pmatrix}, & (48) \\
\frac{\partial}{\partial r}\begin{pmatrix} \boldsymbol{e}_\lambda \\ \boldsymbol{e}_\phi \\ \boldsymbol{e}_r \end{pmatrix} &= 0. & (49)
\end{aligned}
$$

A.3　微分演算子

$$
\begin{aligned}
\nabla &= \boldsymbol{e}_\lambda \frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda} + \boldsymbol{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi} + \boldsymbol{e}_r \frac{\partial}{\partial r}, & (50) \\
\nabla\cdot\boldsymbol{v} &= \left(\boldsymbol{e}_\lambda \frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda} + \boldsymbol{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi} + \boldsymbol{e}_r \frac{\partial}{\partial r}\right)\cdot(v_\lambda \boldsymbol{e}_\lambda + v_\phi \boldsymbol{e}_\phi + v_r \boldsymbol{e}_r) & \\
&= \frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda}v_\lambda + \frac{1}{r\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi v_\phi) + \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 v_r), & (51) \\
\nabla\times\boldsymbol{v} &= \left(\boldsymbol{e}_\lambda \frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda} + \boldsymbol{e}_\phi \frac{1}{r}\frac{\partial}{\partial\phi} + \boldsymbol{e}_r \frac{\partial}{\partial r}\right)\times(v_\lambda \boldsymbol{e}_\lambda + v_\phi \boldsymbol{e}_\phi + v_r \boldsymbol{e}_r) & \\
&= \boldsymbol{e}_\lambda \frac{1}{r}\left[\frac{\partial}{\partial\phi}v_r - \frac{\partial}{\partial r}(r v_\phi)\right] & \\
&\quad + \boldsymbol{e}_\phi \frac{1}{r}\left[\frac{\partial}{\partial r}(r v_\lambda) - \frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}v_r\right] & \\
&\quad + \boldsymbol{e}_r \frac{1}{r\cos\phi}\left[\frac{\partial}{\partial\lambda}v_\phi - \frac{\partial}{\partial\phi}(\cos\phi v_\lambda)\right], & (52) \\
\nabla^2 f &= \nabla\cdot\nabla f & \\
&= \left[\frac{1}{r^2\cos^2\phi}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{r^2\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\phi}\right) + \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right)\right] f & \\
&= \left[\frac{1}{r^2\cos^2\phi}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2} - \frac{\tan\phi}{r^2}\frac{\partial}{\partial\phi} + \frac{1}{r}\frac{\partial^2}{\partial r^2}r\right] f & \\
&= \frac{1}{r^2}\left[\frac{\partial}{\partial\mu}(1-\mu^2)\frac{\partial}{\partial\mu} + \frac{1}{1-\mu^2}\frac{\partial^2}{\partial\lambda^2}\right] f + \frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r}f, & \\
&\quad (\mu = \sin\phi), & (53) \\
\boldsymbol{v}\cdot\nabla\boldsymbol{A} &= \left(\frac{v_\lambda}{r\cos\phi}\frac{\partial}{\partial\lambda} + \frac{v_\phi}{r}\frac{\partial}{\partial\phi} + v_r\frac{\partial}{\partial r}\right)\cdot(A_\lambda \boldsymbol{e}_\lambda + A_\phi \boldsymbol{e}_\phi + A_r \boldsymbol{e}_r) &
\end{aligned}
$$

$$
\begin{aligned}
&= e_\lambda \left[\boldsymbol{v} \cdot \nabla A_\lambda - \frac{\tan\phi}{r} v_\lambda A_\phi + \frac{1}{r} v_\lambda A_r \right] \\
&\quad + e_\phi \left[\boldsymbol{v} \cdot \nabla A_\phi + \frac{\tan\phi}{r} v_\lambda A_\lambda + \frac{1}{r} v_\phi A_r \right] \\
&\quad + e_r \left[\boldsymbol{v} \cdot \nabla A_r - \frac{1}{r} v_\lambda A_\lambda - \frac{1}{r} v_\phi A_\phi \right], \qquad (54)
\end{aligned}
$$

$$
\begin{aligned}
\nabla \times \nabla \times \boldsymbol{v} &= e_\lambda \left[-\frac{1}{r^2}\frac{\partial}{\partial\phi}\left(\frac{1}{\cos\phi}\frac{\partial\cos\phi v_\lambda}{\partial\phi}\right) - \frac{1}{r}\frac{\partial^2 r v_\lambda}{\partial r^2} + \frac{1}{r^2}\frac{\partial}{\partial\phi}\left(\frac{1}{\cos\phi}\frac{\partial v_\phi}{\partial\lambda}\right) \right. \\
&\qquad \left. + \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{\cos\phi}\frac{\partial v_r}{\partial\lambda}\right) \right] \\
&\quad + e_\phi \left[-\frac{1}{r}\frac{\partial^2 r v_\phi}{\partial r^2} - \frac{1}{r^2\cos^2\phi}\frac{\partial^2 v_\phi}{\partial\lambda^2} + \frac{1}{r}\frac{\partial}{\partial r}\frac{\partial v_r}{\partial\phi} + \frac{1}{r^2\cos^2\phi}\frac{\partial}{\partial\lambda}\frac{\partial\cos\phi v_\lambda}{\partial\phi} \right] \\
&\quad + e_r \left[-\frac{1}{r^2\cos^2\phi}\frac{\partial^2 v_r}{\partial\lambda^2} - \frac{1}{r^2\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial v_r}{\partial\phi}\right) + \frac{1}{r^2\cos\phi}\frac{\partial}{\partial\lambda}\frac{\partial r v_\lambda}{\partial r} \right. \\
&\qquad \left. + \frac{1}{r^2\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial r v_\phi}{\partial r}\right) \right] \qquad (55)
\end{aligned}
$$

注意: 球面上の座標を張ることに存在する条件である. スカラー関数が座標上の極 ($\phi = \pm\pi/2$) で特異でない条件をつけることが必要になる場合が多い. 例えば

$$
\left.\frac{\partial^n f}{\partial\lambda^n}\right|_{\phi=\pm\frac{\pi}{2}} = 0 \qquad (n = 1, 2, 3, \cdots), \qquad (56)
$$

$$
\left.\frac{\partial}{\partial\phi}\int f e^{-im\lambda}\, d\lambda\right|_{\phi=\pm\frac{\pi}{2}} = 0 \qquad (m \neq 1). \qquad (57)
$$

後者は波数 1 のもの以外は極において緯度方向 (ϕ 方向) 微分を持ってはいけないということである[2].

A.4 球面上の面積分

面積分:

$$
\int dS = \int_{-\pi/2}^{\pi/2} d\phi \int_0^{2\pi} r^2 \cos\phi d\lambda = \int_{-1}^{1} d\mu \int_0^{2\pi} r^2 d\lambda. \qquad (58)
$$

部分積分. A, B を球面上の滑らかな関数とすれば

$$
\int A \nabla^2 B dS = -\int \nabla A \cdot \nabla B dS = \int \nabla^2 A \cdot B dS, \qquad (59)
$$

$$
\int A \frac{\partial B}{\partial\lambda} dS = -\int \frac{\partial A}{\partial\lambda} B dS. \qquad (60)
$$

---

[2] f が C^{-n} 級である, すなわち, ∂_x, ∂_y の作用に関して滑らかであることを要請すればこのような条件が適宜得られる. よく使うのは $\nabla^2 f$ を勘定してみることである.

A.5　連続の式

連続の式の極座標表示は

$$\frac{\partial \rho}{\partial t} + \frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda}\rho v_\lambda + \frac{1}{r\cos\phi}\frac{\partial}{\partial\phi}\cos\phi\rho v_\phi + \frac{1}{r^2}\frac{\partial}{\partial\phi}r^2\rho v_r \;=\; 0. \tag{61}$$

A.6　ナビエストークス方程式

回転系のナビエストークス方程式は

$$\frac{\partial}{\partial t}\boldsymbol{v} + \boldsymbol{v}\cdot\nabla\boldsymbol{v} + 2\boldsymbol{\Omega}\times\boldsymbol{v} \;=\; -\frac{1}{\rho}\nabla p + \nu\nabla^2\boldsymbol{v}, \tag{62}$$

ただし

$$\nabla^2\boldsymbol{v} \;\equiv\; \underline{\nabla(\nabla\cdot\boldsymbol{v}) - \nabla\times\nabla\times\boldsymbol{v}.} \tag{63}$$

これを極座標表示すると

$$\begin{aligned}
&\frac{\partial}{\partial t}v_\lambda + \boldsymbol{v}\cdot\nabla v_\lambda + \frac{1}{r}(v_r v_\lambda - v_\phi v_\lambda\tan\phi) - 2\Omega\sin\phi v_\phi + 2\Omega\cos\phi v_r \\
&\quad = -\frac{1}{\rho}\frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda}p \\
&\qquad +\nu\left[\frac{1}{r^2\cos^2\phi}\frac{\partial^2 v_\lambda}{\partial\lambda^2} + \frac{1}{r^2}\frac{\partial^2 v_\lambda}{\partial\phi^2} - \frac{\tan\phi}{r^2}\frac{\partial v_\lambda}{\partial\phi} + \frac{1}{r}\frac{\partial^2 r v_\lambda}{\partial r^2}\right. \\
&\qquad\quad \left. + \frac{2}{r^2\cos\phi}\frac{\partial v_r}{\partial\lambda} - \frac{2\sin\phi}{r^2\cos^2\phi}\frac{\partial v_\phi}{\partial\lambda} - \frac{v_\lambda}{r^2\cos^2\phi}\right],
\end{aligned} \tag{64}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}v_\phi + \boldsymbol{v}\cdot\nabla v_\phi + \frac{1}{r}(v_r v_\phi + v_\lambda^2\tan\phi) + 2\Omega\sin\phi v_\lambda \\
&\quad = -\frac{1}{\rho}\frac{1}{r}\frac{\partial}{\partial\phi}p \\
&\qquad +\nu\left[\frac{1}{r^2\cos^2\phi}\frac{\partial^2 v_\phi}{\partial\lambda^2} + \frac{1}{r^2}\frac{\partial^2 v_\phi}{\partial\phi^2} - \frac{\tan\phi}{r^2}\frac{\partial v_\phi}{\partial\phi} + \frac{1}{r}\frac{\partial^2 r v_\phi}{\partial r^2}\right. \\
&\qquad\quad \left. + \frac{2\sin\phi}{r^2\cos^2\phi}\frac{\partial v_\lambda}{\partial\lambda} + \frac{2}{r^2}\frac{\partial v_r}{\partial\phi} - \frac{v_\phi}{r^2\cos^2\phi}\right],
\end{aligned} \tag{65}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}v_r + \boldsymbol{v}\cdot\nabla v_r - \frac{1}{r}(v_\lambda^2 + v_\phi^2) - 2\Omega\cos\phi v_\lambda \\
&\quad = -\frac{1}{\rho}\frac{\partial}{\partial r}p \\
&\qquad +\nu\left[\frac{1}{r^2\cos^2\phi}\frac{\partial^2 v_r}{\partial\lambda^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial\phi^2} - \frac{\tan\phi}{r^2}\frac{\partial v_r}{\partial\phi} + \frac{1}{r}\frac{\partial^2 r v_r}{\partial r^2}\right. \\
&\qquad\quad \left. - \frac{2}{r^2}\frac{\partial v_\phi}{\partial\phi} + \frac{2\tan\phi v_\phi}{r^2} - \frac{2}{r^2\cos\phi}\frac{\partial v_\lambda}{\partial\lambda} - \frac{2v_r}{r^2}\right].
\end{aligned} \tag{66}$$

ただし, 極座標の極は系の回転軸と一致するように選んである.

A.7 参考：歪テンソル

$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\nabla p + \nabla\cdot e$

曲線直行座標系では歪みテンソルは

$$e_{\xi\eta} = e_\xi\cdot(e_\eta\cdot\nabla)v + e_\eta\cdot(e_\xi\cdot\nabla)v \tag{67}$$

で与えられる. ξ, η に λ, ϕ, r を代入して

$2\nabla(\nabla\cdot v) - \nabla\times\nabla\times v$

$$\begin{aligned}
e_{\lambda\lambda} &= \frac{2}{r\cos\phi}\frac{\partial v_\lambda}{\partial\lambda} - \frac{2v_\phi\tan\phi}{r} + 2\frac{v_r}{r}, && (68)\\
e_{\phi\phi} &= \frac{2}{r}\frac{\partial v_\phi}{\partial\phi} + 2\frac{v_r}{r}, && (69)\\
e_{rr} &= 2\frac{\partial v_r}{\partial r}, && (70)\\
e_{\lambda\phi} &= \frac{1}{r\cos\phi}\frac{\partial v_\phi}{\partial\lambda} + \frac{\cos\phi}{r}\frac{\partial}{\partial\phi}\frac{v_\lambda}{\cos\phi}, && (71)\\
e_{\phi r} &= \frac{1}{r}\frac{\partial v_r}{\partial\phi} + r\frac{\partial}{\partial r}\frac{v_\phi}{r}, && (72)\\
e_{r\lambda} &= r\frac{\partial}{\partial r}\frac{v_\lambda}{r} + \frac{1}{r\cos\phi}\frac{\partial v_r}{\partial\lambda}. && (73)
\end{aligned}$$

なお, この歪みテンソルに $\nabla\cdot$ を作用しても先のナビエストークス方程式の粘性項の表現は得られないことに注意. 書き下すと

$$\left(e_\lambda\frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda} + e_\phi\frac{1}{r}\frac{\partial}{\partial\phi} + e_r\frac{\partial}{\partial r}\right)\cdot(e_{\xi\eta}e_\xi\otimes e_\eta). \tag{74}$$

ただし, 同じ添え字が繰り返し出てきた時には縮約ととる. また

$$e_\zeta\cdot e_\xi\otimes e_\eta \equiv e_\xi\delta_{\zeta\eta} \tag{75}$$

である. この計算を実行すると

$$\begin{aligned}
&\left(e_\lambda\frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda} + e_\phi\frac{1}{r}\frac{\partial}{\partial\phi} + e_r\frac{\partial}{\partial r}\right)(e_{\xi\eta}e_\xi\otimes e_\eta)\\
&= e_\lambda\left\{\frac{1}{r\cos\phi}\left[\frac{\partial e_{\lambda\lambda}}{\partial\lambda} - 2\sin\phi e_{\lambda\phi} + 2\cos\phi e_{r\lambda}\right] + \frac{1}{r}\left[\frac{\partial e_{\lambda\phi}}{\partial\phi} + e_{r\lambda}\right] + \frac{\partial e_{r\lambda}}{\partial r}\right\}\\
&\quad + e_\phi\left\{\frac{1}{r\cos\phi}\left[\frac{\partial e_{\lambda\phi}}{\partial\lambda} - \sin\phi e_{\phi\phi} + \cos\phi e_{\phi r} + \sin\phi e_{\lambda\lambda}\right] + \frac{1}{r}\left[\frac{\partial e_{\phi\phi}}{\partial\phi} + 2e_{\phi r}\right] + \frac{\partial e_{\phi r}}{\partial r}\right\}\\
&\quad + e_r\left\{\frac{1}{r\cos\phi}\left[\frac{\partial e_{r\lambda}}{\partial\lambda} - \sin\phi e_{\phi r} + \cos\phi e_{rr} - \cos\phi e_{\lambda\lambda}\right] + \frac{1}{r}\left[\frac{\partial e_{\phi r}}{\partial\phi} - e_{\phi\phi} + e_{rr}\right] + \frac{\partial e_{rr}}{\partial r}\right\}\\
&= e_\lambda\left\{\frac{1}{r^2\cos^2\phi}\frac{\partial^2 v_\lambda}{\partial\lambda^2} + \frac{1}{r^2}\frac{\partial^2 v_\lambda}{\partial\phi^2} - \frac{\tan\phi}{r^2}\frac{\partial v_\lambda}{\partial\phi} + \frac{1}{r}\frac{\partial^2 r v_\lambda}{\partial r^2}\right.\\
&\qquad + \frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda}\left[\frac{1}{r\cos\phi}\frac{\partial v_\lambda}{\partial\lambda} + \frac{1}{r}\frac{\partial v_\lambda}{\partial\phi} - \frac{\tan\phi v_\lambda}{r} + \frac{1}{r}\frac{\partial r v_\lambda}{\partial r}\right]
\end{aligned}$$

$$
\begin{aligned}
&+\frac{2}{r^2\cos\phi}\frac{\partial v_r}{\partial\lambda}-\frac{2\sin\phi}{r^2\cos^2\phi}\frac{\partial v_\phi}{\partial\lambda}-\frac{v_\lambda}{r^2\cos^2\phi}\Bigg\} \\
+e_\phi&\Bigg\{\frac{1}{r^2\cos^2\phi}\frac{\partial^2 v_\phi}{\partial\lambda^2}+\frac{1}{r^2}\frac{\partial^2 v_\phi}{\partial\phi^2}-\frac{\tan\phi}{r^2}\frac{\partial v_\phi}{\partial\phi}+\frac{1}{r}\frac{\partial^2 r v_\phi}{\partial r^2} \\
&+\frac{1}{r}\frac{\partial}{\partial\phi}\left[\frac{1}{r\cos\phi}\frac{\partial v_\lambda}{\partial\lambda}+\frac{1}{r}\frac{\partial v_\lambda}{\partial\phi}-\frac{\tan\phi v_\lambda}{r}+\frac{1}{r}\frac{\partial r v_\lambda}{\partial r}\right] \\
&+\frac{2\sin\phi}{r^2\cos^2\phi}\frac{\partial v_\lambda}{\partial\lambda}+\frac{2}{r^2}\frac{\partial v_r}{\partial\phi}-\frac{v_\phi}{r^2\cos^2\phi}\Bigg\} \\
+e_r&\Bigg\{\frac{1}{r^2\cos^2\phi}\frac{\partial^2 v_r}{\partial\lambda^2}+\frac{1}{r^2}\frac{\partial^2 v_r}{\partial\phi^2}-\frac{\tan\phi}{r^2}\frac{\partial v_r}{\partial\phi}+\frac{1}{r}\frac{\partial^2 r v_r}{\partial r^2} \\
&+\frac{\partial}{\partial r}\left[\frac{1}{r\cos\phi}\frac{\partial v_\lambda}{\partial\lambda}+\frac{1}{r}\frac{\partial v_\lambda}{\partial\phi}-\frac{\tan\phi v_\lambda}{r}+\frac{1}{r}\frac{\partial r v_\lambda}{\partial r}\right] \\
&-\frac{2}{r^2}\frac{\partial v_\phi}{\partial\phi}+\frac{2\tan\phi v_\phi}{r^2}-\frac{2}{r^2\cos\phi}\frac{\partial v_\lambda}{\partial\lambda}-\frac{2v_r}{r^2}\Bigg\}.
\end{aligned}
\tag{76}
$$

$\nabla\cdot\boldsymbol{v}=0$ の時のみ [ ] の項が消えて一致する. そもそも, 通常の表式 (64) ~ (66) は $\nabla\cdot\boldsymbol{v}=0$ のもとで導出されているものであるから, その手続きと互換性を保つとすれば, 粘性項の表現は$-\nu\nabla\times\nabla\times\boldsymbol{v}$ であるべきである.

A.8 渦度方程式

渦度方程式は回転系のナビエストークス方程式の表現

$$
\frac{\partial}{\partial t}\boldsymbol{v}+(\boldsymbol{\omega}+2\boldsymbol{\Omega})\times\boldsymbol{v}+\nabla\frac{v^2}{2} \;=\; -\frac{1}{\rho}\nabla p+\nu\nabla^2\boldsymbol{v} \tag{77}
$$

に $\nabla\times$ を作用して

$$
\frac{\partial}{\partial t}\boldsymbol{\omega}+\boldsymbol{v}\cdot\nabla(\boldsymbol{\omega}+2\boldsymbol{\Omega})-\nabla(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\boldsymbol{v} \;=\; \frac{1}{\rho^2}\nabla\rho\times\nabla p-\nu\underline{\nabla\times\nabla\times\boldsymbol{\omega}}. \tag{78}
$$

ただし

$$
\boldsymbol{\omega} \;\equiv\; \nabla\times\boldsymbol{v}. \tag{79}
$$

これを極座標表示すると

$$
\begin{aligned}
&\frac{\partial}{\partial t}\omega_{a\lambda}+\boldsymbol{v}\cdot\nabla\omega_{a\lambda}-\boldsymbol{\omega}_a\cdot\nabla v_\lambda+\frac{1}{r}(\omega_{ar}v_\lambda-\omega_{a\lambda}v_r-\omega_{a\phi}v_\lambda\tan\phi+\omega_{a\lambda}v_\phi\tan\phi) \\
&\qquad=\frac{1}{\rho^2}\frac{1}{r}\left(\frac{\partial\rho}{\partial\phi}\frac{\partial p}{\partial r}-\frac{\partial\rho}{\partial r}\frac{\partial p}{\partial\phi}\right) \\
&\qquad\quad-\nu\left[-\frac{1}{r^2}\frac{\partial}{\partial\phi}\left(\frac{1}{\cos\phi}\frac{\partial\cos\phi\,\omega_{a\lambda}}{\partial\phi}\right)-\frac{1}{r}\frac{\partial^2 r\omega_{a\lambda}}{\partial r^2}+\frac{1}{r^2}\frac{\partial}{\partial\phi}\left(\frac{1}{\cos\phi}\frac{\partial\omega_{a\phi}}{\partial\lambda}\right)\right. \\
&\qquad\qquad\left.+\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{1}{\cos\phi}\frac{\partial\omega_{ar}}{\partial\lambda}\right)\right],
\end{aligned}
\tag{80}
$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\omega_{a\phi} + \boldsymbol{v}\cdot\nabla\omega_{a\phi} - \boldsymbol{\omega}_a\cdot\nabla v_\phi + \frac{1}{r}(\omega_{ar}v_\phi - \omega_{a\phi}v_r) \\
&\qquad = \frac{1}{\rho^2}\frac{1}{r\cos\phi}\left(\frac{\partial\rho}{\partial r}\frac{\partial p}{\partial\lambda} - \frac{\partial\rho}{\partial\lambda}\frac{\partial p}{\partial r}\right) \\
&\qquad\quad -\nu\left[-\frac{1}{r}\frac{\partial^2 r\omega_{a\phi}}{\partial r^2} - \frac{1}{r^2\cos^2\phi}\frac{\partial^2\omega_{a\phi}}{\partial\lambda^2} + \frac{1}{r}\frac{\partial}{\partial r}\frac{\partial\omega_{ar}}{\partial\phi} + \frac{1}{r^2\cos^2\phi}\frac{\partial}{\partial\lambda}\frac{\partial\cos\phi\omega_{a\lambda}}{\partial\phi}\right],
\end{aligned} \tag{81}$$

$$\begin{aligned}
&\frac{\partial}{\partial t}\omega_{ar} + \boldsymbol{v}\cdot\nabla\omega_{ar} - \boldsymbol{\omega}_a\cdot\nabla v_r \\
&\qquad = \frac{1}{\rho^2}\frac{1}{r^2\cos\phi}\left(\frac{\partial\rho}{\partial\lambda}\frac{\partial p}{\partial\phi} - \frac{\partial\rho}{\partial\phi}\frac{\partial p}{\partial\lambda}\right) \\
&\qquad\quad -\nu\left[-\frac{1}{r^2\cos^2\phi}\frac{\partial^2\omega_{ar}}{\partial\lambda^2} - \frac{1}{r^2\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial\omega_{ar}}{\partial\phi}\right) + \frac{1}{r^2\cos\phi}\frac{\partial}{\partial\lambda}\frac{\partial r\omega_{a\lambda}}{\partial r}\right. \\
&\qquad\qquad\qquad \left. + \frac{1}{r^2\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial r\omega_{a\phi}}{\partial r}\right)\right].
\end{aligned} \tag{82}$$

ただし

$$\begin{aligned}
\boldsymbol{\omega}_a &\equiv \nabla\times\boldsymbol{v} + 2\boldsymbol{\Omega} \\
&= \boldsymbol{e}_\lambda\frac{1}{r}\left[\frac{\partial}{\partial\phi}v_r - \frac{\partial}{\partial r}(rv_\phi)\right] \\
&\quad +\boldsymbol{e}_\phi\left[\frac{1}{r}\frac{\partial}{\partial r}(rv_\lambda) - \frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda}v_r + 2\Omega\cos\phi\right] \\
&\quad +\boldsymbol{e}_r\left[\frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda}v_\phi - \frac{1}{r\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi v_\lambda) + 2\Omega\sin\phi\right].
\end{aligned} \tag{83}$$

B　非発散 2 次元球面方程式系の導出

世界は回転系にある非発散ナビエストークス流体として記述されるものとする. 密度は一定 ($\rho = \rho_0$) であり, 運動は球面に拘束されている.

B.1　球面への拘束

球面への拘束条件は次のように与えることにする.

$$v_r = 0, \tag{84}$$

$$v_\lambda, v_\phi \propto r. \tag{85}$$

B.2　連続の式

密度一定, 球面拘束のもとでは, 連続の式 (61) は

$$\frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}v_\lambda + \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\cos\phi v_\phi = 0 \tag{86}$$

となる.

B.3　ナビエストークスの式

密度一定, 球面拘束のもとでは, ナビエストークスの式 (64) ~ (66) は

$$\begin{aligned}
\frac{\partial}{\partial t}v_\lambda + \frac{v_\lambda}{r\cos\phi}\frac{\partial v_\lambda}{\partial\lambda} + \frac{v_\phi}{r}\frac{\partial v_\lambda}{\partial\phi} - \frac{\tan\phi}{r}v_\lambda v_\phi - 2\Omega\sin\phi v_\phi
&= -\frac{1}{\rho_0}\frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda}p \\
&\quad +\nu\left[\frac{1}{r^2\cos^2\phi}\frac{\partial^2 v_\lambda}{\partial\lambda^2} + \frac{1}{r^2}\frac{\partial^2 v_\lambda}{\partial\phi^2} - \frac{\tan\phi}{r^2}\frac{\partial v_\lambda}{\partial\phi} + \frac{2v_\lambda}{r^2}\right. \\
&\quad \left. -\frac{2\sin\phi}{r^2\cos^2\phi}\frac{\partial v_\phi}{\partial\lambda} - \frac{v_\lambda}{r^2\cos^2\phi}\right],
\end{aligned} \tag{87}$$

$$\begin{aligned}
\frac{\partial}{\partial t}v_\phi + \frac{v_\lambda}{r\cos\phi}\frac{\partial v_\phi}{\partial\lambda} + \frac{v_\phi}{r}\frac{\partial v_\phi}{\partial\phi} + \frac{\tan\phi}{r}v_\lambda^2 + 2\Omega\sin\phi v_\lambda
&= -\frac{1}{\rho_0}\frac{1}{r}\frac{\partial}{\partial\phi}p \\
&\quad +\nu\left[\frac{1}{r^2\cos^2\phi}\frac{\partial^2 v_\phi}{\partial\lambda^2} + \frac{1}{r^2}\frac{\partial^2 v_\phi}{\partial\phi^2} - \frac{\tan\phi}{r^2}\frac{\partial v_\phi}{\partial\phi} + \frac{2v_\phi}{r^2}\right. \\
&\quad \left. +\frac{2\sin\phi}{r^2\cos^2\phi}\frac{\partial v_\lambda}{\partial\lambda} - \frac{v_\phi}{r^2\cos^2\phi}\right]
\end{aligned} \tag{88}$$

$$\tag{89}$$

となる.

B.4 渦度方程式

密度一定, 球面拘束のもとでは, 渦度方程式の動径成分 (82) は

$$\begin{aligned}\frac{\partial}{\partial t}\omega_{ar} + \frac{v_\lambda}{r\cos\phi}\frac{\partial \omega_{ar}}{\partial \lambda} + \frac{v_\phi}{r}\frac{\partial \omega_{ar}}{\partial \phi} \\ = \nu\left[\frac{1}{r^2\cos^2\phi}\frac{\partial^2\omega_{ar}}{\partial\lambda^2} + \frac{1}{r^2\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial\omega_{ar}}{\partial\phi}\right) + \frac{2\omega_{ar}}{r^2}\right]\end{aligned} \qquad (90)$$

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となる. 粘性項の最後の項は, 球面拘束の元での渦度が

$$\begin{aligned}\omega_a &= -e_\lambda 2\frac{v_\phi}{r} + e_\phi\left[2\frac{v_\lambda}{r} + 2\Omega\cos\phi\right] \\ &\quad + e_r\left[\frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda}v_\phi - \frac{1}{r\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi v_\lambda) + 2\Omega\sin\phi\right]\end{aligned} \qquad (91)$$

であることを用い, (82) に代入したものである.

B.5 流線関数を用いた表現

球面上で非発散であるので流線関数 ψ が次のように導入できる:

$$-\frac{1}{r}\frac{\partial\psi}{\partial\phi} \equiv v_\lambda, \qquad (92)$$

$$\frac{1}{r\cos\phi}\frac{\partial\psi}{\partial\lambda} \equiv v_\phi. \qquad (93)$$

渦度の動径成分は

$$\begin{aligned}\omega_r &= \frac{1}{r\cos\phi}\frac{\partial}{\partial\lambda}v_\phi - \frac{1}{r\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi v_\lambda) \\ &= \left[\frac{1}{r^2\cos^2\phi}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{r^2\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\phi}\right)\right]\psi \\ &= \frac{1}{r^2}\nabla_h^2\psi.\end{aligned} \qquad (94)$$

∇_h^2 は半径 1 の球面上の 2 次元ラプラシアンである.

流線関数を用いれば渦度方程式の動径成分は

$$\frac{\partial}{\partial t}\nabla_h^2\psi - \frac{1}{r^2\cos\phi}\frac{\partial\psi}{\partial\phi}\frac{\partial\nabla_h^2\psi}{\partial\lambda} + \frac{1}{r^2\cos\phi}\frac{\partial\psi}{\partial\lambda}\frac{\partial\nabla_h^2\psi + 2\Omega\sin\phi}{\partial\psi} = \nu\frac{1}{r^2}(\nabla_h^2+2)\nabla_h^2\psi \qquad (95)$$

となる.

B.6　まとめ

球面系での方程式は, v_λ を u, v_ϕ を v, そして, r を拘束している球の半径 a と書き変えて

$$\begin{aligned}
\frac{\partial}{\partial t}u + \frac{u}{a\cos\phi}\frac{\partial u}{\partial\lambda} + \frac{v}{a}\frac{\partial u}{\partial\phi} - \frac{\tan\phi uv}{a} - 2\Omega\sin\phi v &= -\frac{1}{\rho_0}\frac{1}{a\cos\phi}\frac{\partial}{\partial\lambda}p \\
&\quad +\nu\left[\frac{1}{a^2}(\nabla_h^2+2)u - \frac{2\sin\phi}{a^2\cos^2\phi}\frac{\partial v}{\partial\lambda} - \frac{u}{a^2\cos^2\phi}\right], \quad (96)\\
\frac{\partial}{\partial t}v + \frac{u}{a\cos\phi}\frac{\partial v}{\partial\lambda} + \frac{v}{a}\frac{\partial v}{\partial\phi} + \frac{\tan\phi u^2}{a} + 2\Omega\sin\phi u &= -\frac{1}{\rho_0}\frac{1}{a}\frac{\partial}{\partial\phi}p \\
&\quad +\nu\left[\frac{1}{a^2}(\nabla_h^2+2)v + \frac{2\sin\phi}{a^2\cos^2\phi}\frac{\partial u}{\partial\lambda} - \frac{v}{a^2\cos^2\phi}\right], \quad (97)\\
\frac{1}{\cos\phi}\frac{\partial}{\partial\lambda}u + \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\cos\phi v &= 0. \quad (98)
\end{aligned}$$

流線関数を用いた渦度方程式は

$$\frac{\partial}{\partial t}\nabla_h^2\psi - \frac{1}{a^2\cos\phi}\frac{\partial\psi}{\partial\phi}\frac{\partial\nabla_h^2\psi}{\partial\lambda} + \frac{1}{a^2\cos\phi}\frac{\partial\psi}{\partial\lambda}\frac{\partial\nabla_h^2\psi}{\partial\psi} + \frac{2\Omega}{a^2}\frac{\partial\psi}{\partial\lambda} = \nu\frac{1}{a^2}(\nabla_h^2+2)\nabla_h^2\psi, \quad (99)$$

あるいは

$$\frac{\partial}{\partial t}\zeta + \frac{1}{a^2\cos\phi}J(\psi,\zeta+2\Omega\sin\phi) = \nu\frac{1}{a^2}(\nabla_h^2+2)\zeta, \quad (100)$$

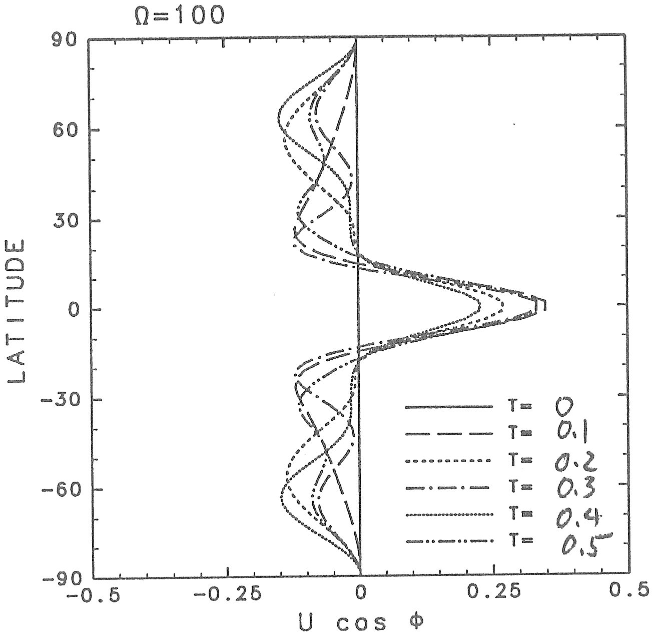
あるいは

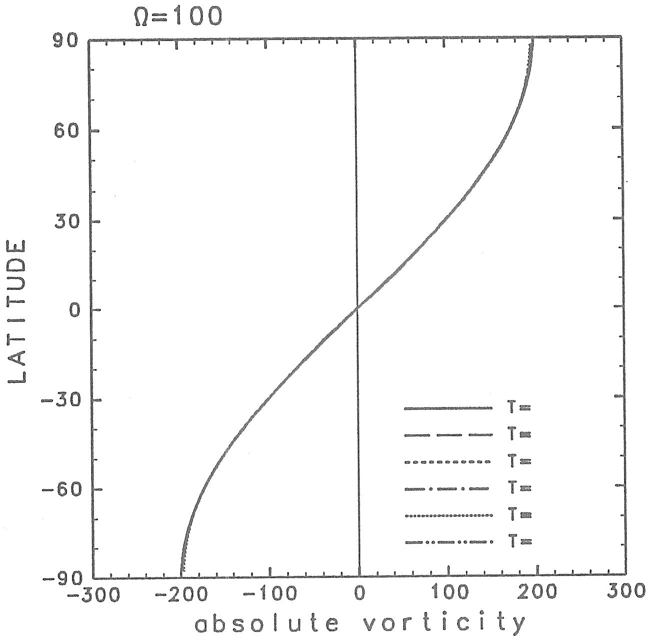
$$\frac{\partial}{\partial t}q + \frac{1}{a^2\cos\phi}J(\psi,q) = \nu\frac{1}{a^2}(\nabla_h^2+2)q. \quad (101)$$

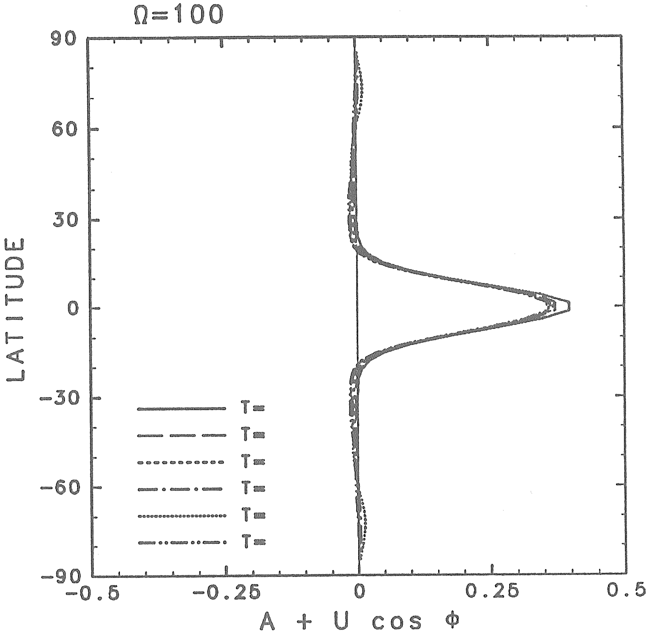
ただし,

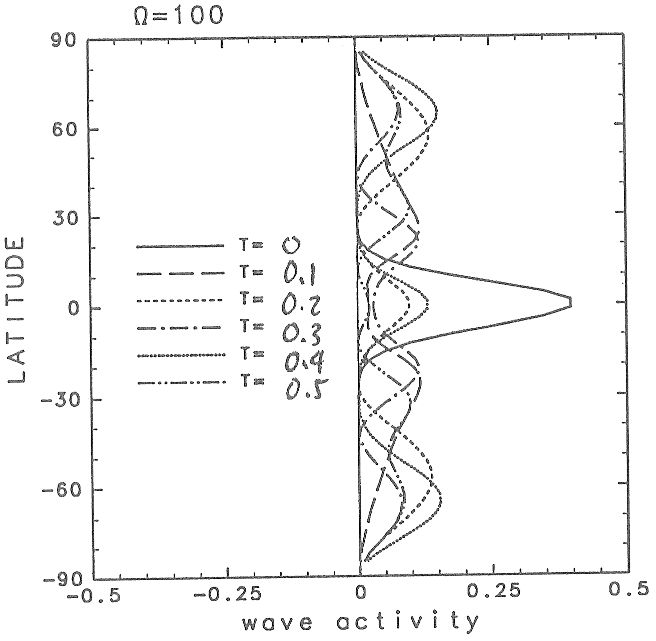
$$\begin{aligned}
u &\equiv -\frac{1}{a}\frac{\partial\psi}{\partial\phi}, \quad (102)\\
v &\equiv \frac{1}{a\cos\phi}\frac{\partial\psi}{\partial\lambda}, \quad (103)\\
\zeta &\equiv \frac{1}{a\cos\phi}\frac{\partial}{\partial\lambda}v - \frac{1}{a\cos\phi}\frac{\partial}{\partial\phi}(\cos\phi u)\\
&= \left[\frac{1}{a^2\cos^2\phi}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{a^2\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\phi}\right)\right]\psi\\
&= \frac{1}{r^2}\nabla_h^2\psi, \quad (104)\\
q &\equiv \zeta + 2\Omega\sin\phi, \quad (105)\\
\nabla_h^2 &\equiv \frac{1}{\cos^2\phi}\frac{\partial^2}{\partial\lambda^2} + \frac{1}{\cos\phi}\frac{\partial}{\partial\phi}\left(\cos\phi\frac{\partial}{\partial\phi}\right), \quad (106)\\
J(X,Y) &\equiv \frac{\partial X}{\partial\lambda}\frac{\partial Y}{\partial\phi} - \frac{\partial Y}{\partial\lambda}\frac{\partial X}{\partial\phi}. \quad (107)
\end{aligned}$$

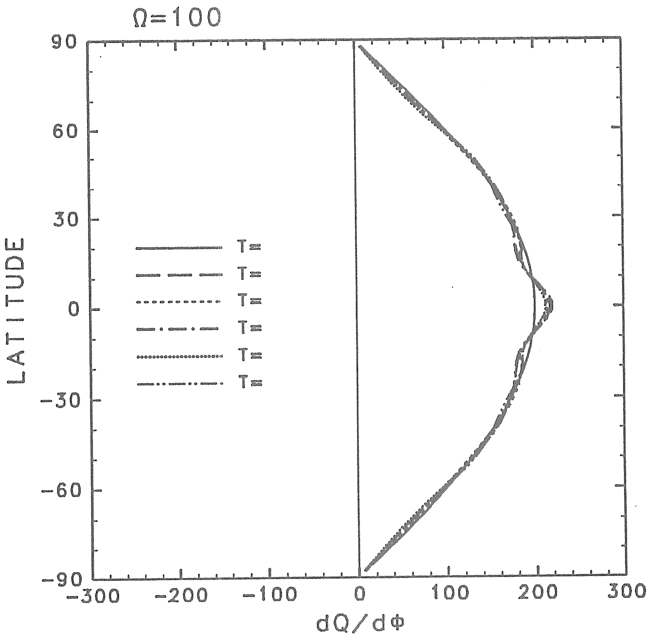
相対渦度の動経成分 ω_r を ζ, 絶対渦度の動経成分 ω_{ar} を q と書き変えた.











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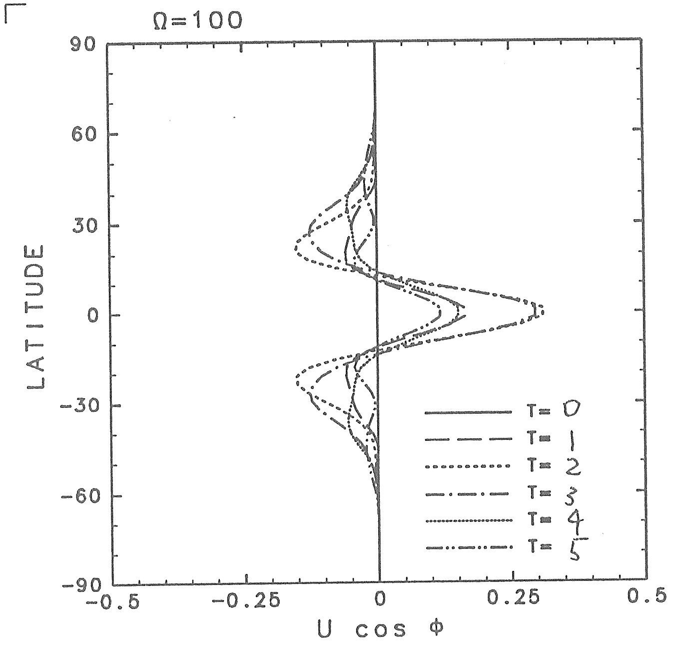
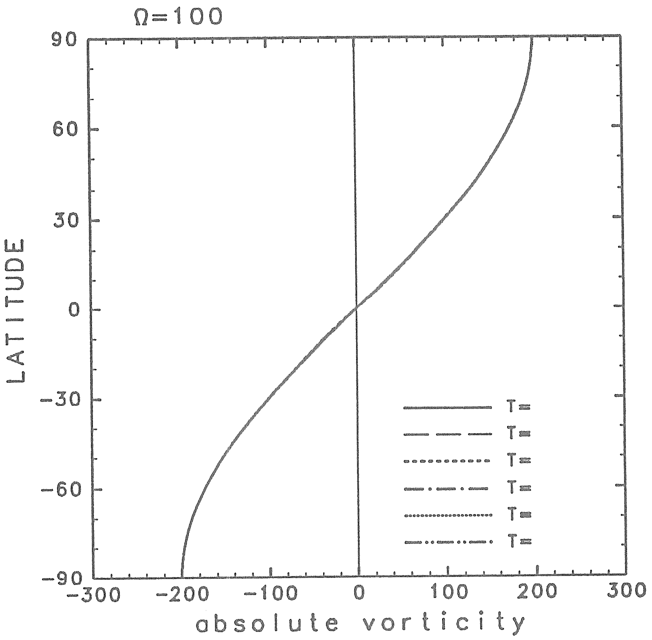
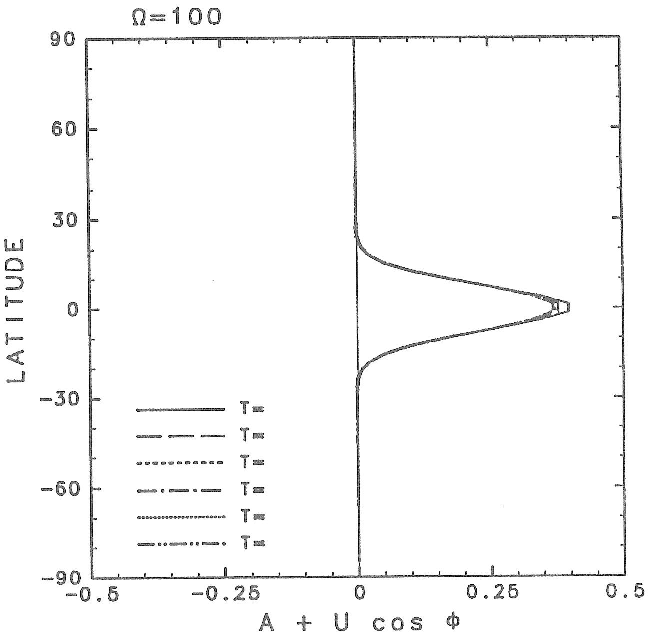
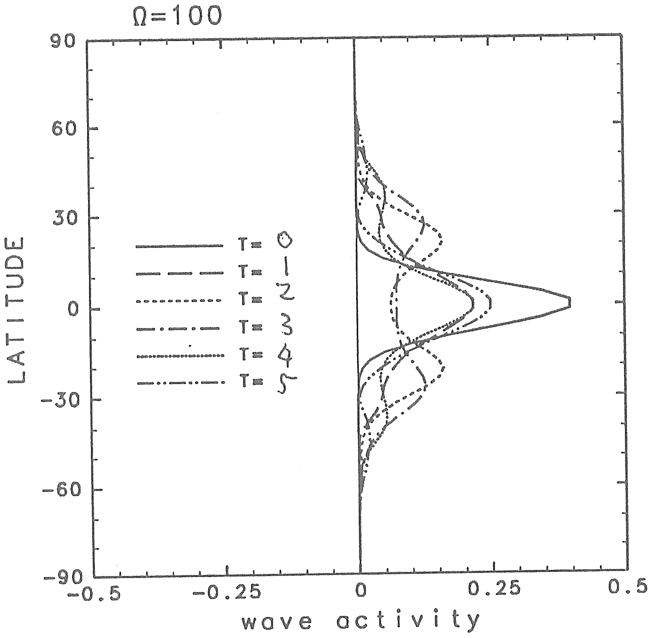
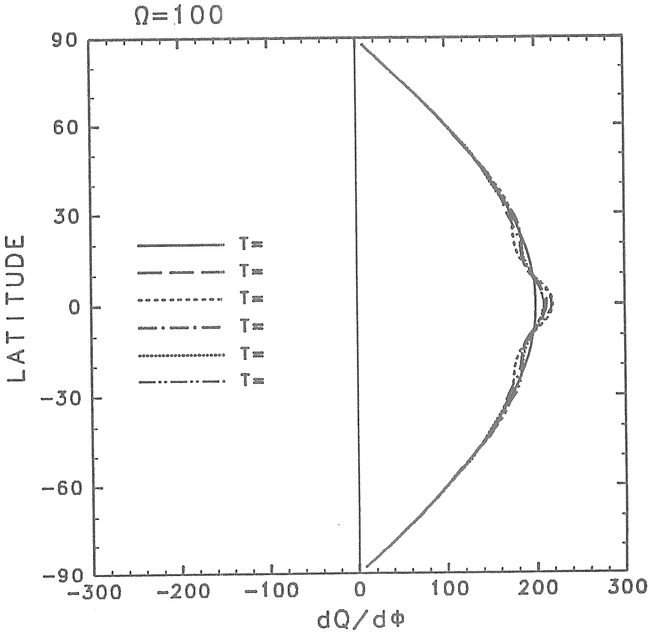
$a= 1.0; \Omega=100.0; \nu_4= 1.0\times10^{-6}$











$a = 1.0; \ \Omega=100.0; \ \nu_4 = 1.0\times10^{-6}$

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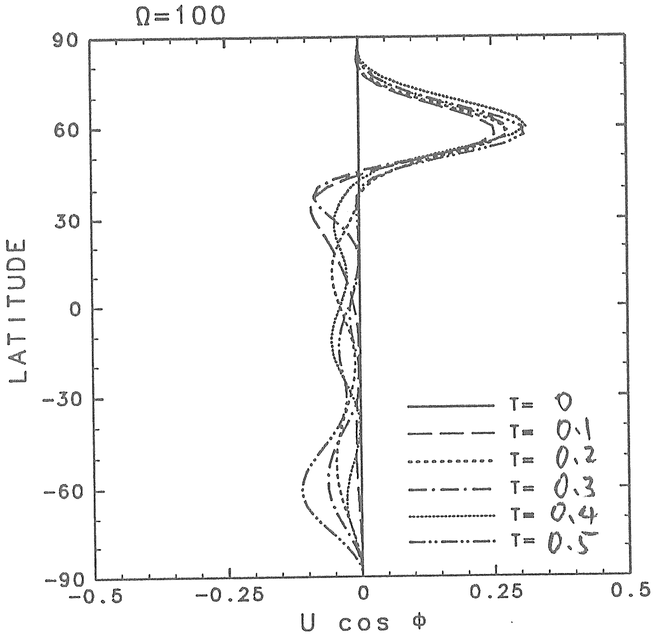

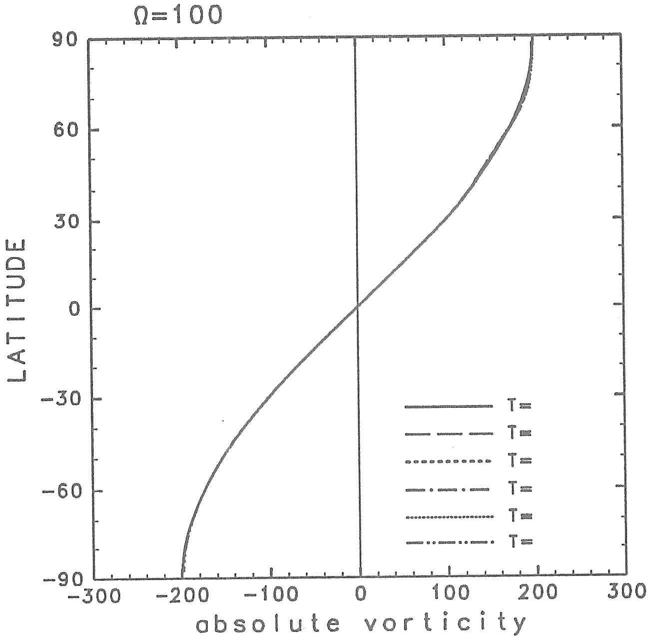

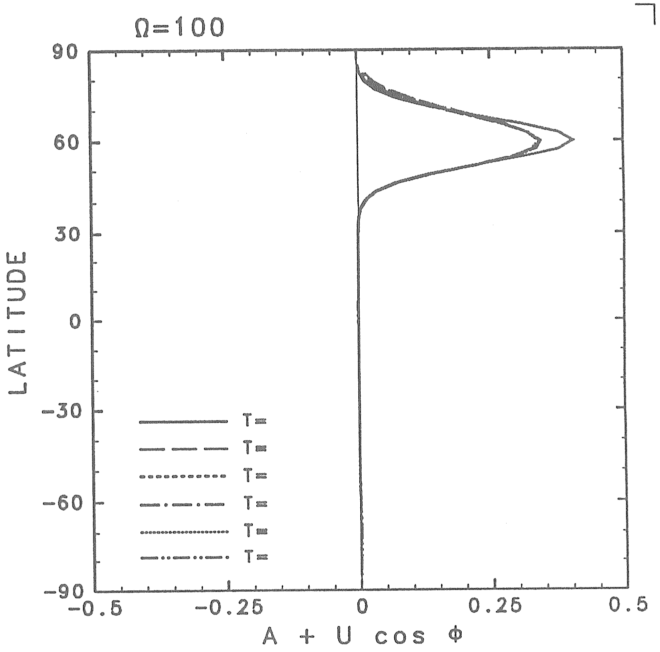

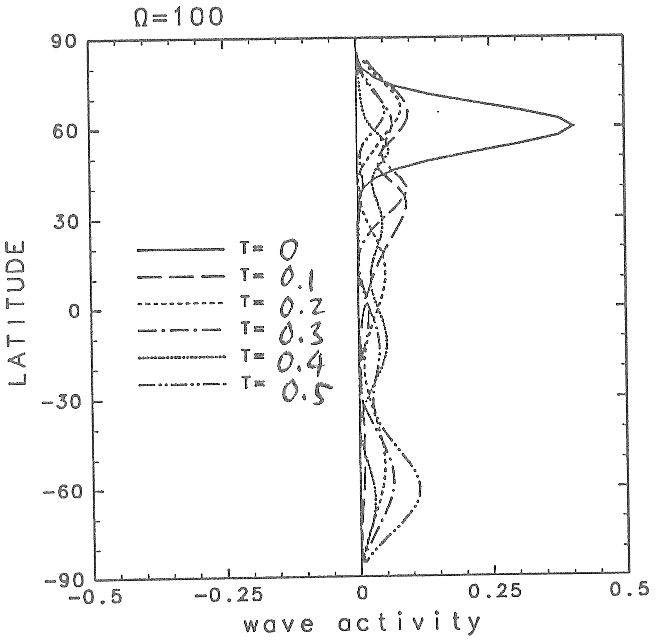

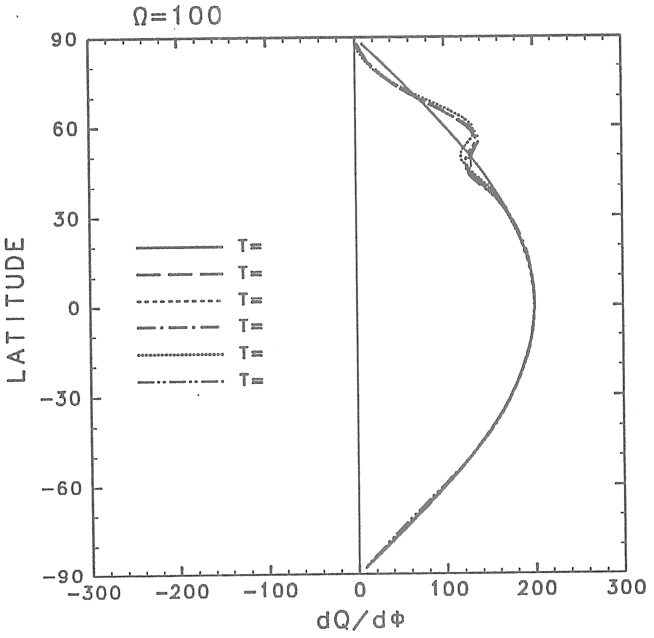

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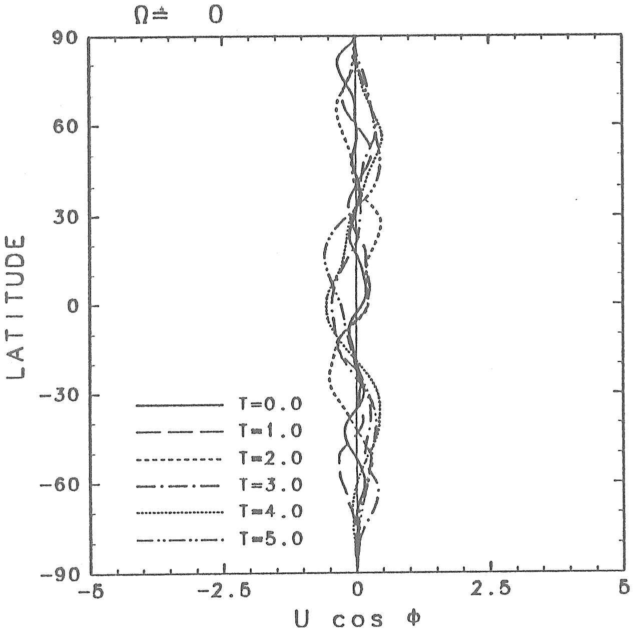
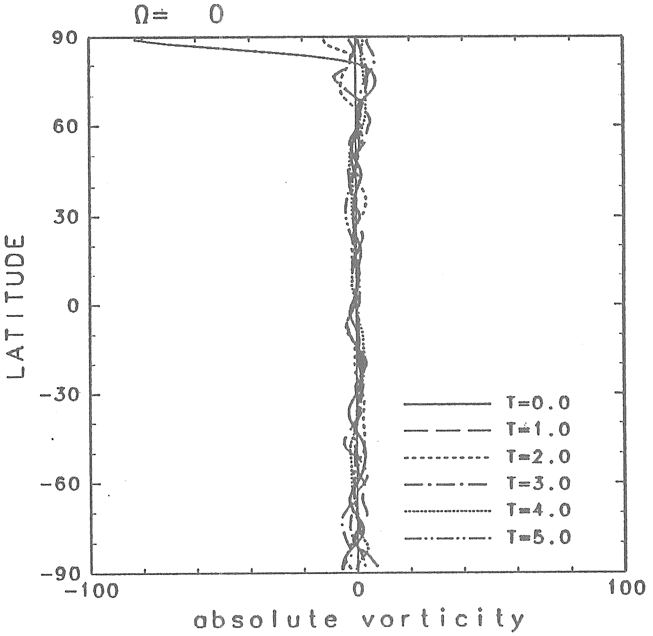
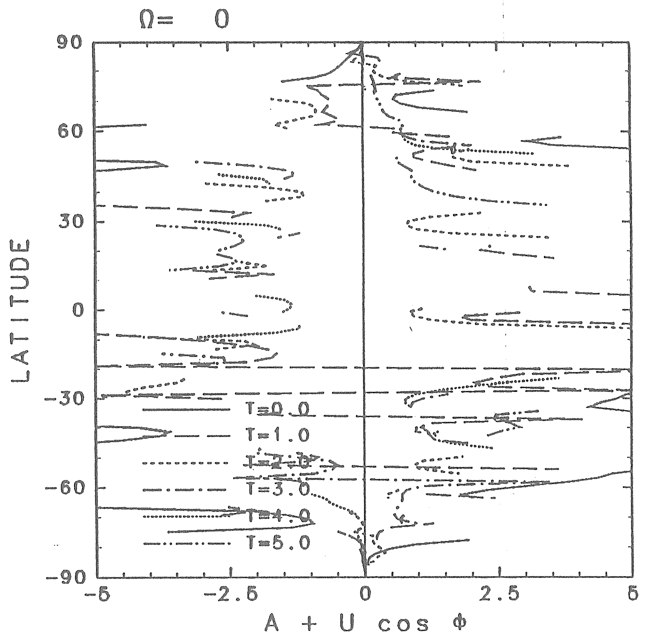
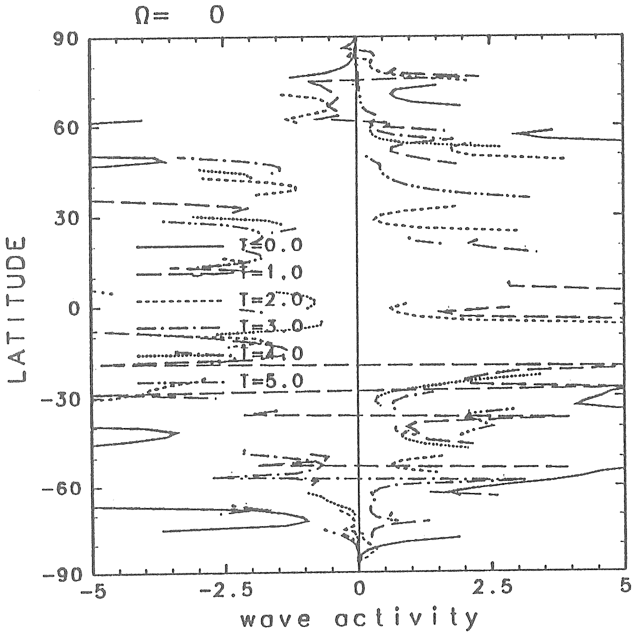
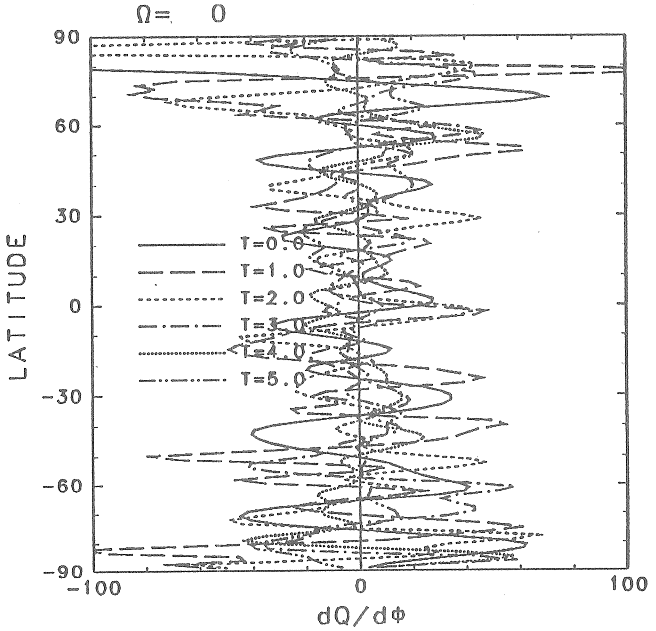
$a= 1.0; \ \Omega=100.0; \ \nu_4= 1.0\times10^{-6}$







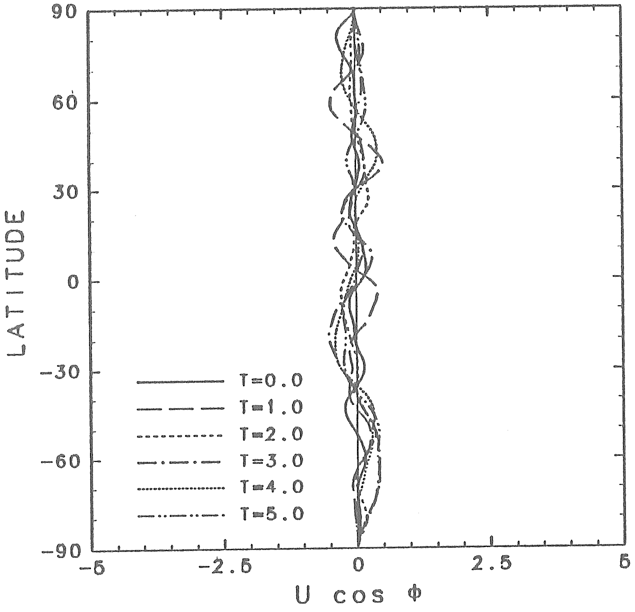




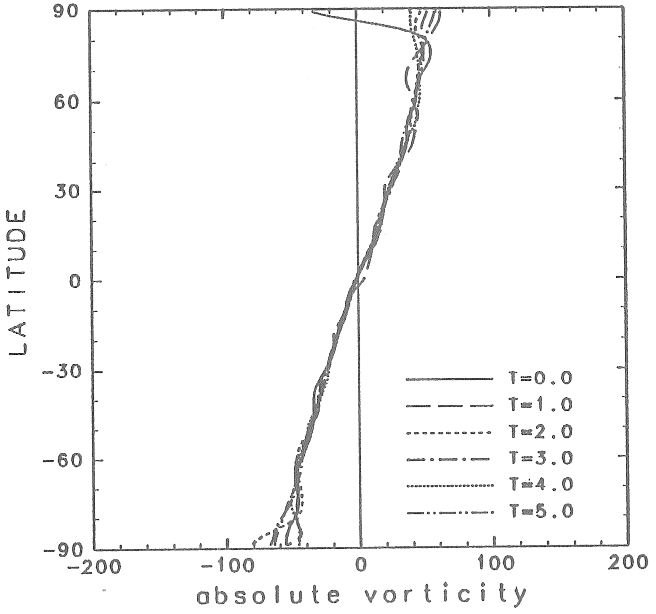
gwzint.f : page 1 : /o6/yoden/rossby/grf/cwzint/az000

$\alpha = 1.0; \ \Omega = \ 0.0; \ \nu_4 = 1.0 \times 10^{-6}$

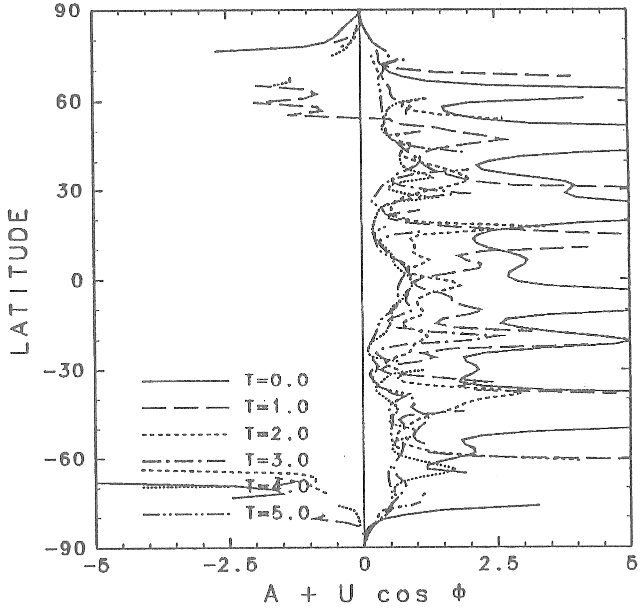
Ω= 25



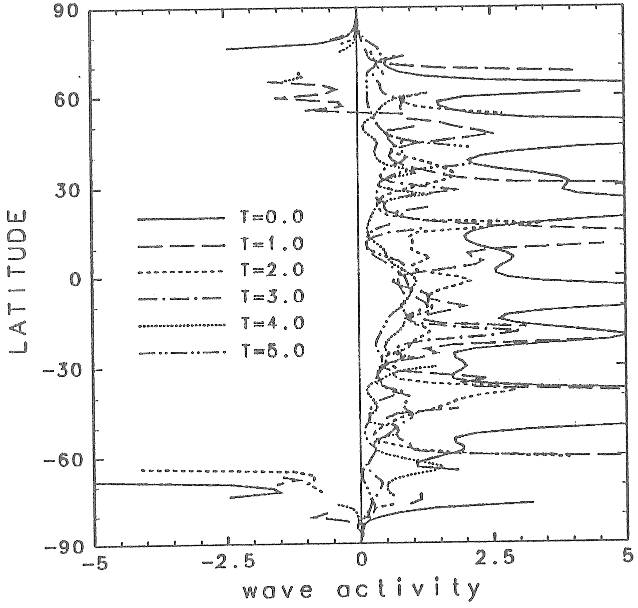
Ω= 25



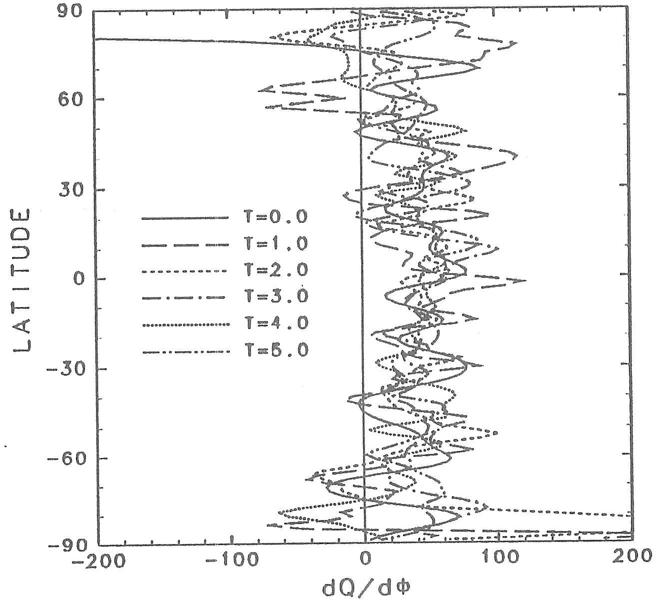
Ω= 25



Ω= 25

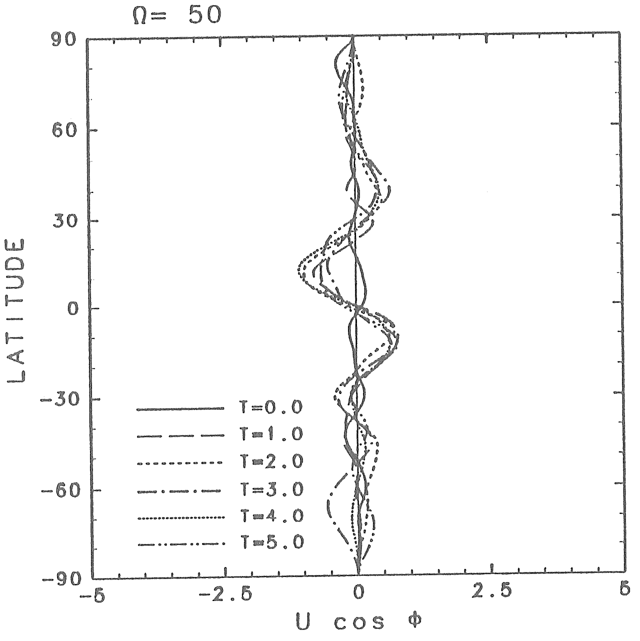


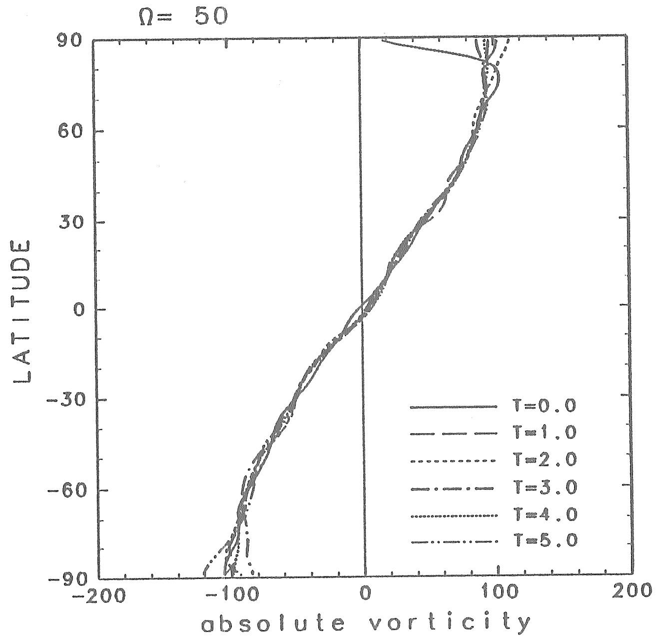
Ω= 25

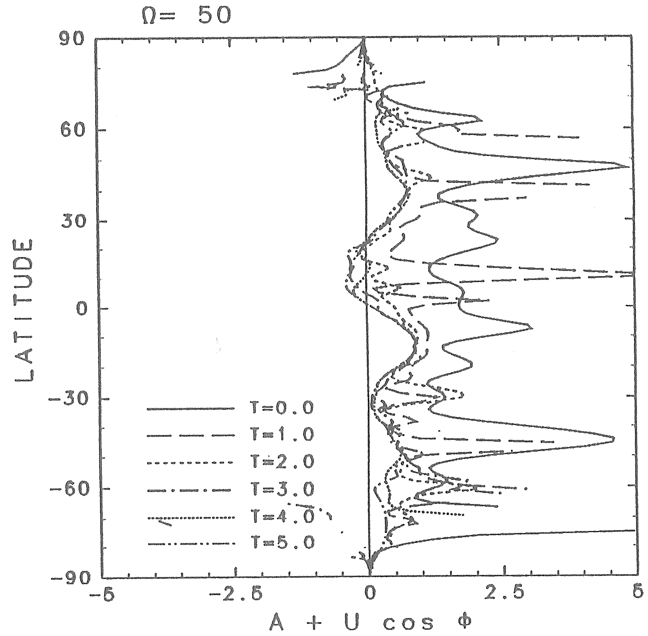


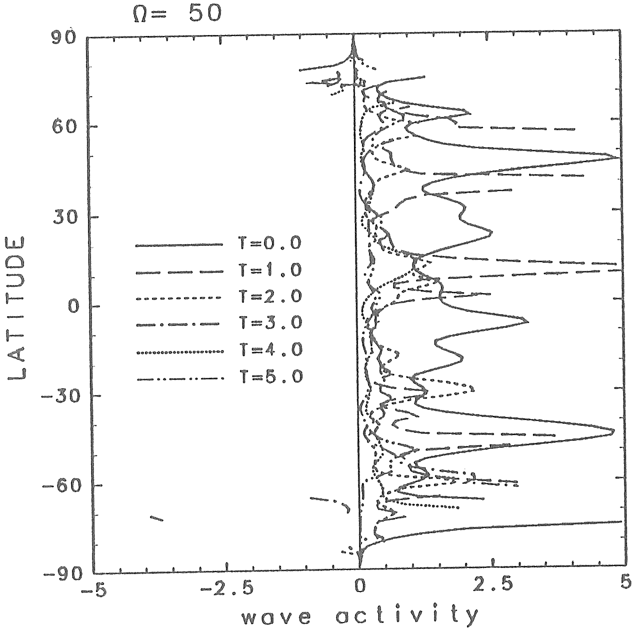
$a= 1.0;\ \Omega= 25.0;\ \nu_4= 1.0\times10^{-6}$

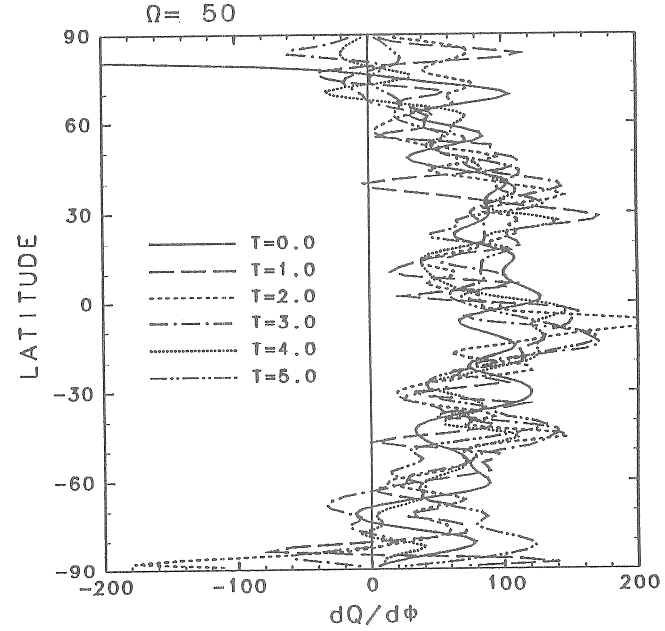
gwzint.f : page 1 : /a6/yoden/rossby/grf/cwzint/az025





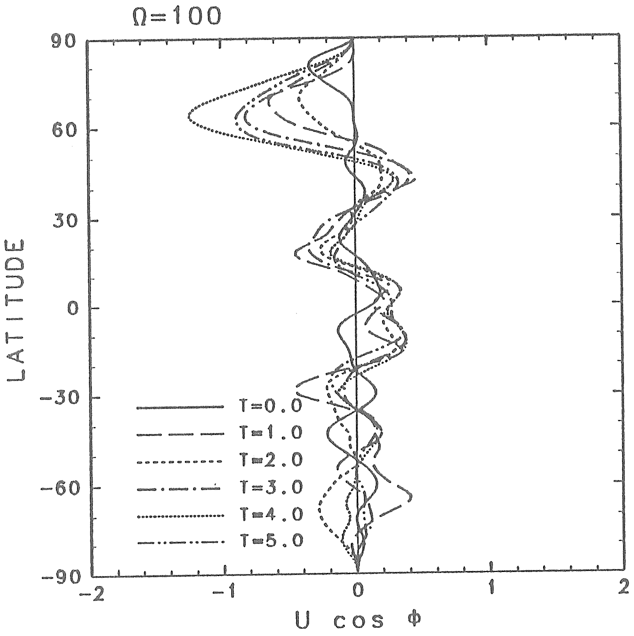


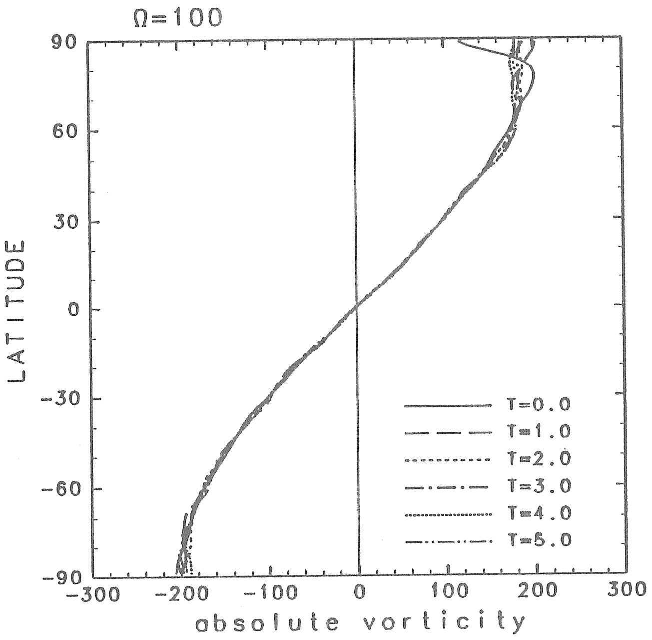


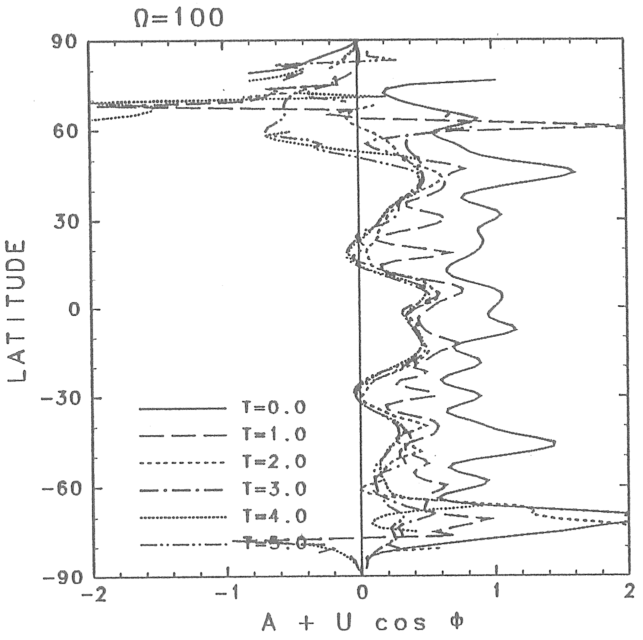


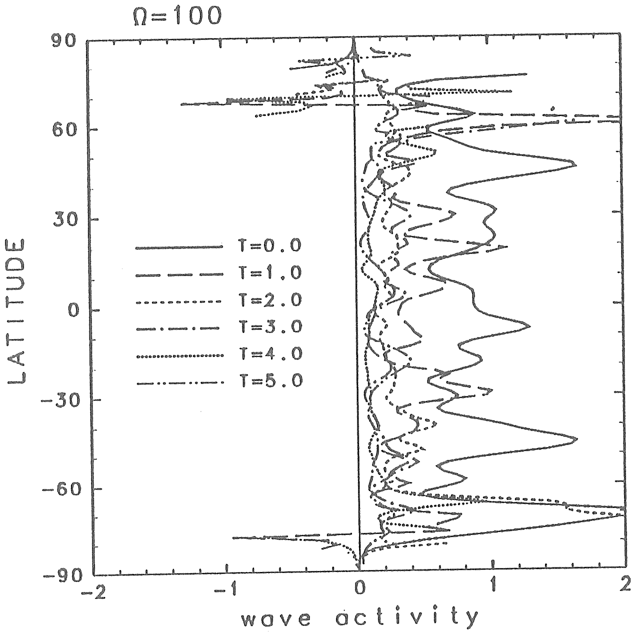
$\alpha= 1.0; \ \Omega= 50.0; \ \nu_4= 1.0\times10^{-6}$

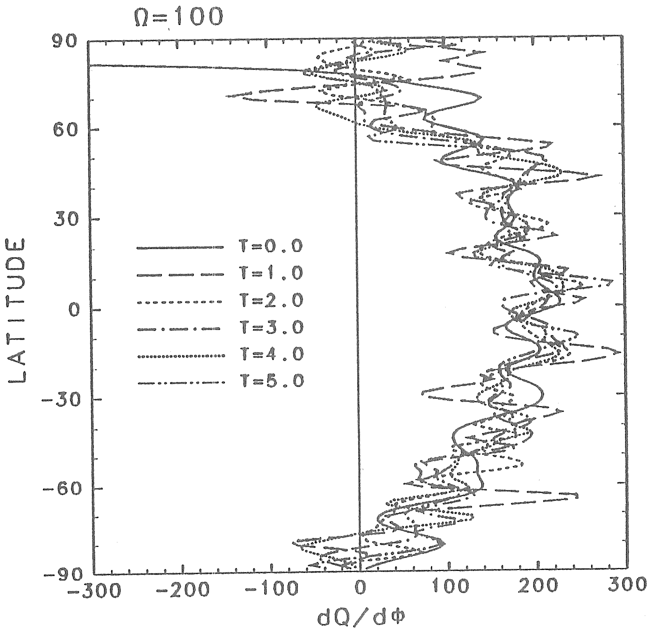
gwzint.f : page 1 : /a6/yoden/rossby/grf/cwzint/oz050



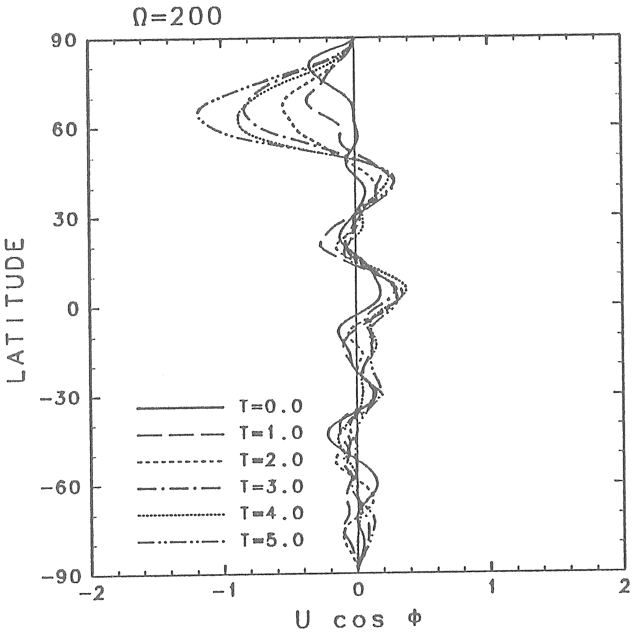


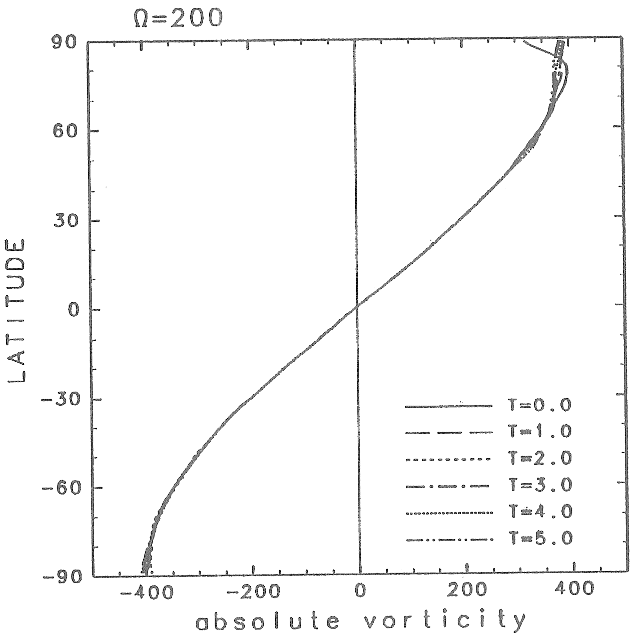


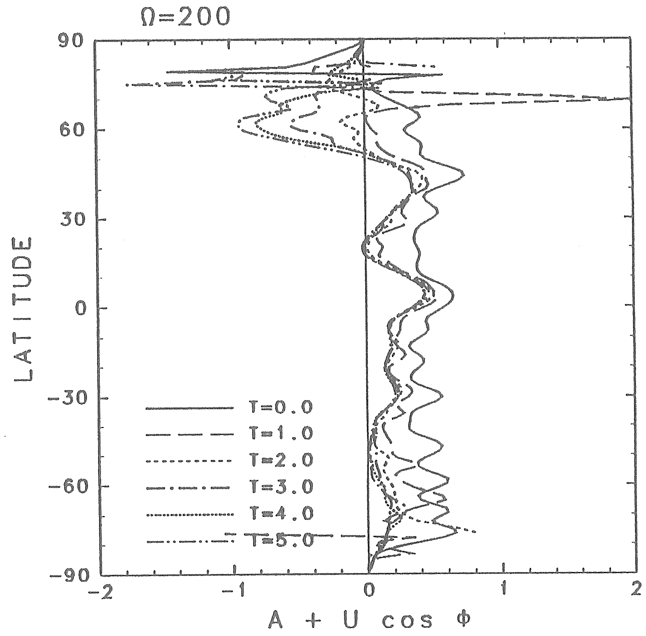


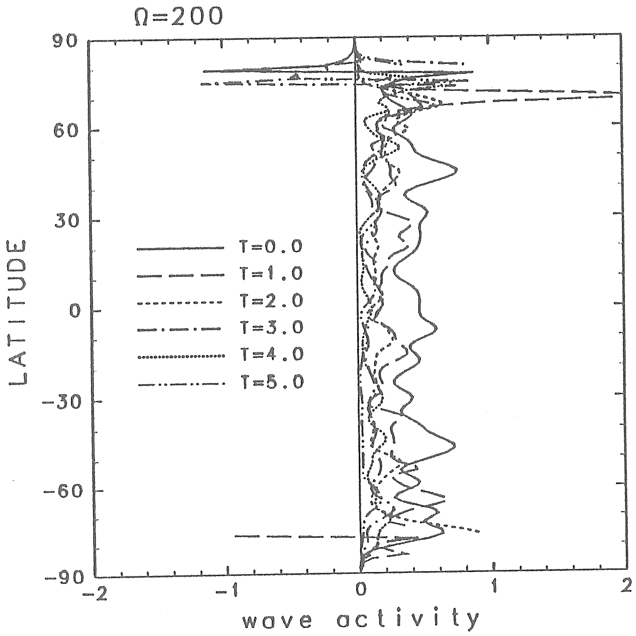


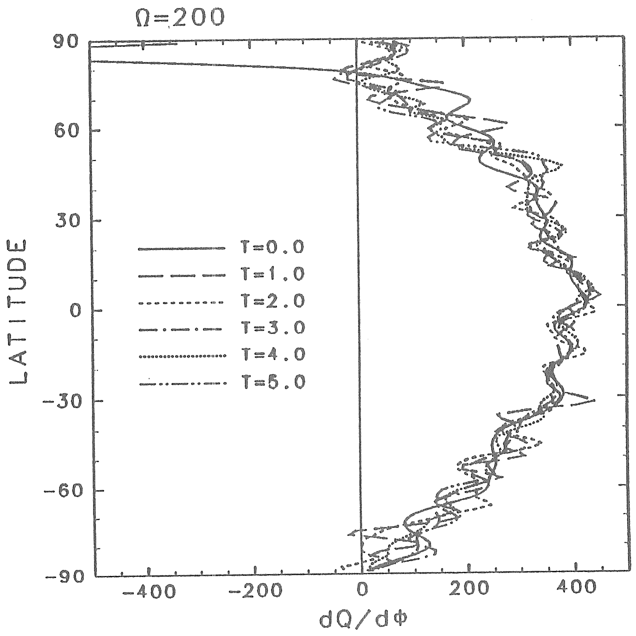
$a= 1.0; \Omega=100.0; \nu_4= 1.0\times10^{-6}$





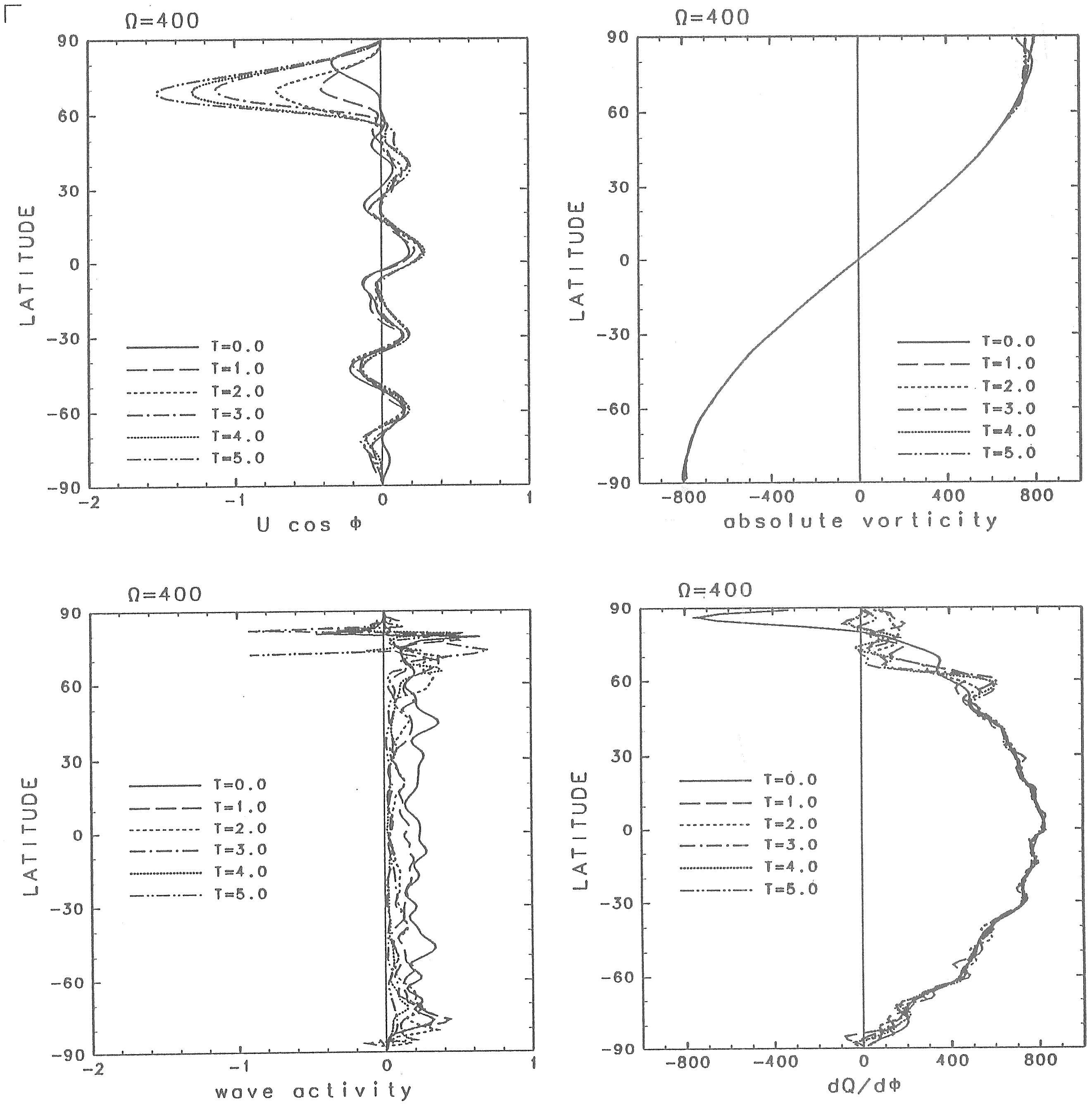






$a= 1.0; \ \Omega=200.0; \ \nu_4= 1.0\times10^{-6}$

gwzint.f : page 1 : /a6/yoden/rossby/grf/cwzint/az200



gwzint.f : page 1 : /a6/yoden/rossby/grf/cwzint/az400

$\alpha = 1.0; \ \Omega=400.0; \ \nu_4 = 1.0 \times 10^{-6}$

Ωのリスト.

	$\Omega^*(s^{-1})$	$a(m)$	$U(m/s)$	Ω
金星	$5\cdot10^{-7}$	$6\cdot10^{6}$	$10\sim100$	$\ll1$
地球	大気 10^{-4}	$6\cdot10^{6}$	$10\sim100$	$60\sim6$
	海 10^{-4}	$6\cdot10^{6}$	1	600
火星	10^{-4}	$3\cdot10^{6}$	10	30
木星	$2\cdot10^{-4}$	$7\cdot10^{7}$	100	100
土星	$2\cdot10^{-4}$	$6\cdot10^{7}$	100	100
太陽	$5\cdot10^{-6}$	$7\cdot10^{8}$	100	30

ロスビー波の 東向(西向) 擬角運動量

⇓

平均角速度の 生成. (東向)

⇓

臨界緯度の発生.

- ・波動伝播の終了
- ・波動の擬運動量の解放

$$U = - \frac{\bar{Q}_{\varphi}(}{k^2+l^2}$$

$\bar{Q}_{\varphi} \sim 2\Omega\cos\varphi$, $k^2+l^2 \sim 10^2$

⇓ $U \sim 1$であるには.

$\Omega = 25$: いたるところ

100 : $\varphi \sim 60°$

400 : $\varphi \sim 80°$

⇓

さらなる東風加速. ?

もう少しロスビー波の特性に注目してみる：弱非線形論

・波のもつ擬運動量 A:

$$\boxed{A \equiv \frac{1}{2}\frac{\overline{\omega'^2}}{\overline{Q}_\varphi}\cdot\cos\varphi}$$

保存則 (WKB)

$$\frac{\partial A}{\partial t} + \nabla\cdot \mathbf{C}_g A = \nu_4 \frac{\overline{\nabla^4\omega'\cdot\omega'}}{\overline{Q}_\varphi}\cos\varphi$$

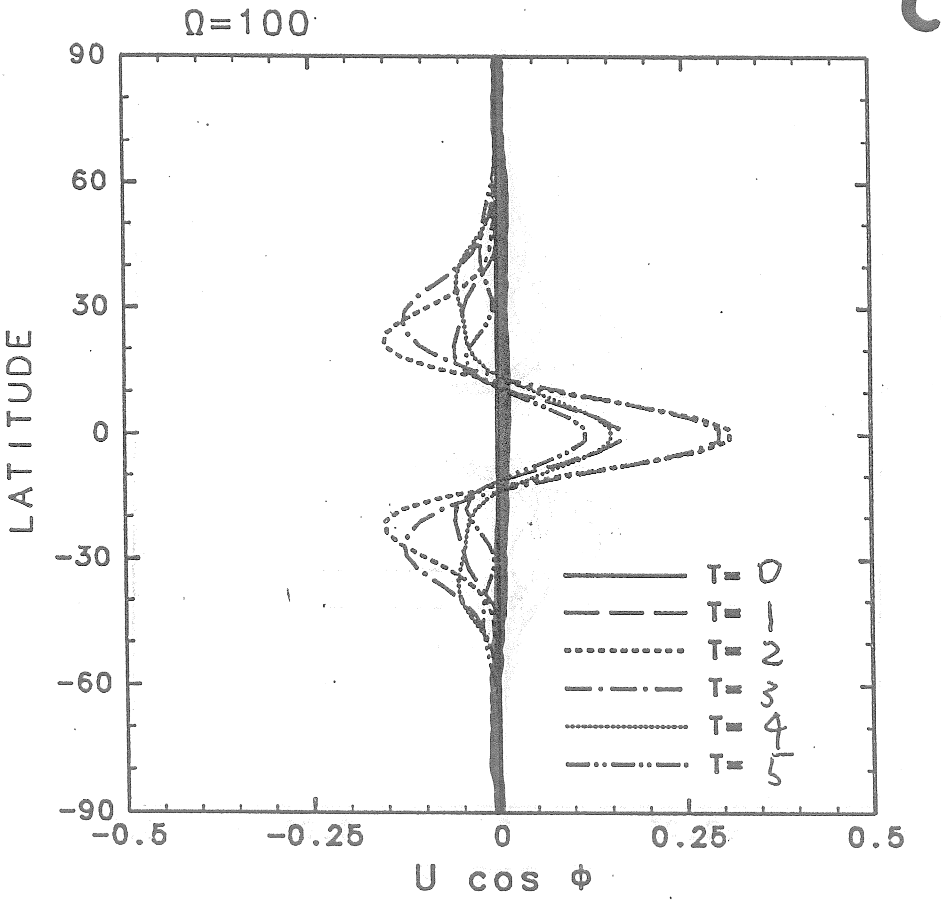
・角運動量とのやりとり．

$$\frac{\partial}{\partial t}\bar{u}\cos\varphi - \nabla \mathbf{C}_g A = 0$$

or

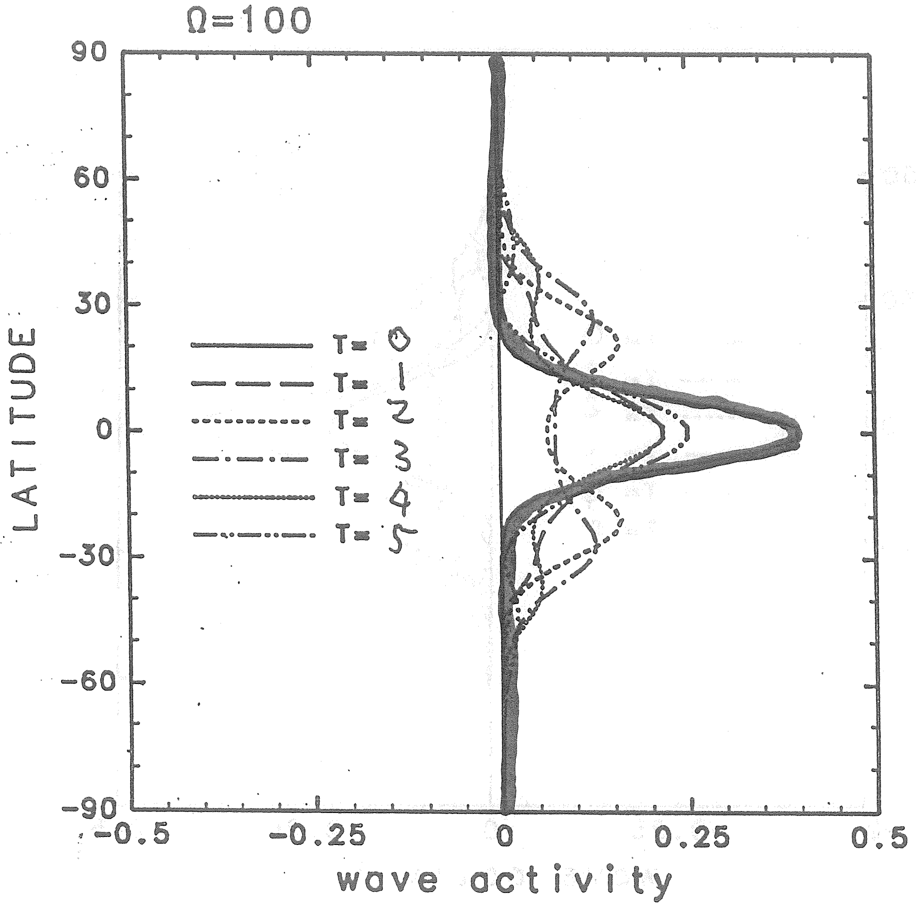
$$\boxed{\frac{\partial}{\partial t}\left[\bar{u}\cos\varphi + A\right] = \nu_4 \frac{\overline{\omega'\nabla^4\omega'}}{\overline{Q}_\varphi}\cos\varphi}$$

t=0~5



角運動量

$\bar{u}\cos\varphi$



A

k=6, Ω=100

ロスビー・波の伝播と 角運動量輸送

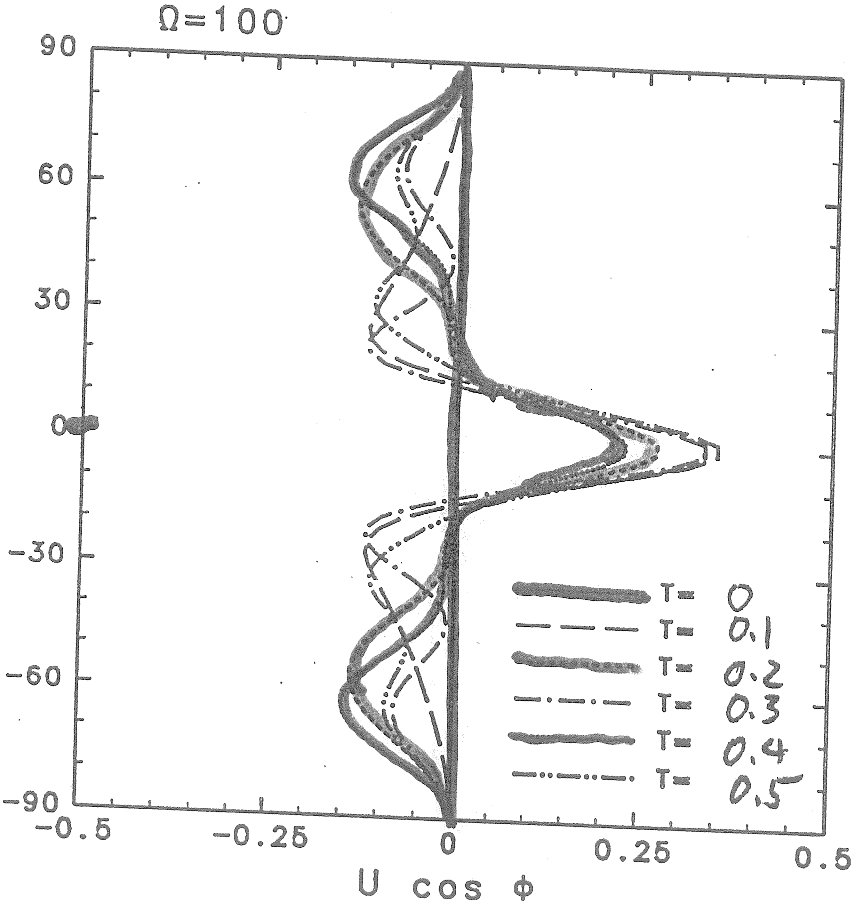
例1

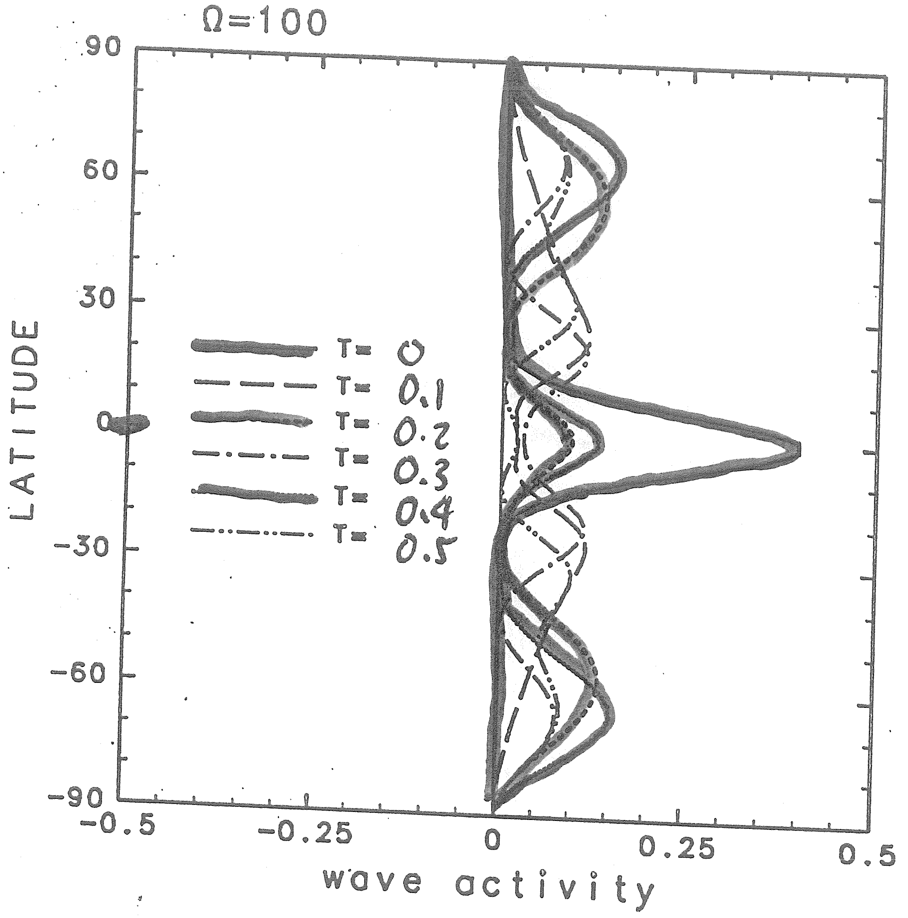
t=0~0.5

t=0 で
赤道上
に
撹乱
をおく。

角運動量

$\bar{u}\cos\varphi$





A

k=1, Ω=100

＜ＦＡＸ通信票＞

92年2月20日(木)

余田 様

FAX NO. ~~075-753-3932~~ 075-721-9249

14枚 お送り致します。（通信票を含む） (A, φ)

※備考：

〒113
東京都文京区弥生 2-11-16
東京大学 理学部 地球惑星物理学教室
林 祥介

☎：０３－３８１２－２１１１
ＥＸＴ．4282/4285
ＦＡＸ：０３－３８１８－３２４７

ＦＡＸの誤送が増えています。
もう一度番号の確認をお願いします

いつまでたってもできない！では困りますので
とりあえず現在やっているメモを送ります。 波の保存量については
カシミール（Haynes & McIntyre）から derive（[illegible]たのではなくて、
旧式の wave-mean でやってあります。 座標系は"[illegible]"ですが"[illegible]"
のほうがよいですか？

という訳で 計算のほうは、

1. weak non-linear
2. packet release
3. ray tracing

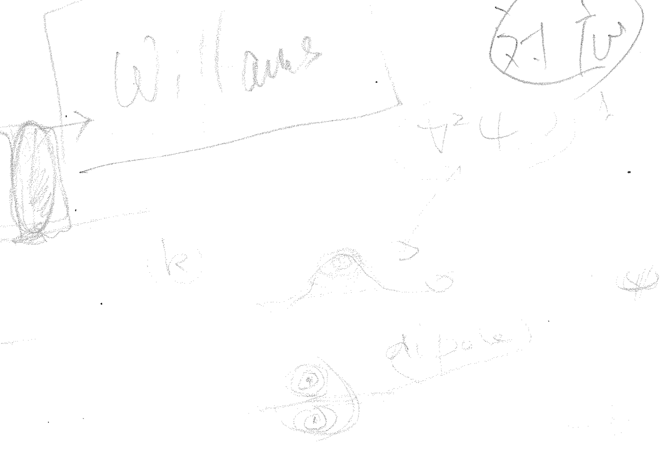
あたりをやってみませんか。

4. すみながし（＝カシミール）— McIntyre の一味がよくやっている例のきれーな絵

をやってみたいのですが……

はやし。

P.S. 山田氏のメールアドレスは何だっけ。



ロスビー波 (2次元非発散球面) D-1

というわけで

これまでの non-linear 版と

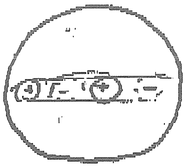
1. 同じ初期値で 弱非線形版 をはしらせる.

$$\text{i.e.}\begin{cases}\dfrac{\partial\zeta'}{\partial t}+J(\overline{\Psi},\zeta')+J(\psi',\overline{\zeta})=F' \\ \dfrac{\partial\overline{\zeta}}{\partial t}+\dfrac{1}{\cos\theta}\dfrac{\partial}{\partial\theta}\left(\overline{\dfrac{\partial\psi'}{\partial\lambda}\zeta'}\right)=\overline{F}\end{cases}$$

ロスビー波 (線形的な) による 角運動量輸送 解釈 が成立する ためには 弱非線形版で おおむね 値が 同じならば すむ!

(QBO みたいなもの…)

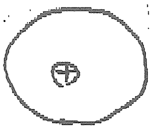
2. "赤道域に 波数 k の 波をおいて 分散を check する



$\overline{u}$ はどうなるか?

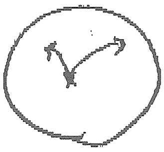
3. 赤道上に "point" source をおいて 分散を check する

$\overline{u}$ はどうなるか



(反対側に ⊖ をおかないとアカン)

4. 波数の計算 (ω, k を適当に与えて)



k, l を ベクトル描いて つないで つなげる.

問題は ω, k の 選択則.

ロスビー波 (2次元非発散モデル)

・質問.

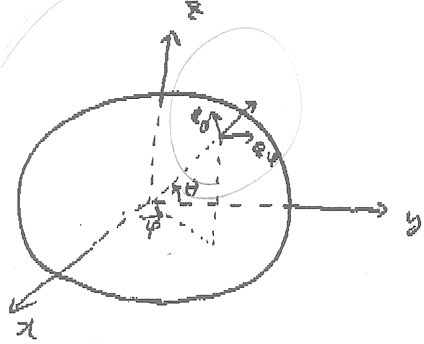
・その昔の Williams の"木星"の計算では
ムジゲン化した Ω 等々はいくつ？
(実験と対応できるのか否か…)

・金星, 地球 火星・木星, 土星, 天王星, 海王星 の "Ω" は？

同じ resolution (parameter) でやった decay experiment はあるか？

ロスビー波 (2次元非発散球面)

基礎方程式 : 半径1の球面で考える. ($\rho=1$ の球面流体)



速度ベクトル

u : 東向き

v : 北向き.

$\boldsymbol{v} = u\boldsymbol{e}_\varphi + v\boldsymbol{e}_\theta$.

連続の式:

$\nabla\cdot\boldsymbol{v} = 0$

i.e.,

$$\frac{1}{\cos\theta}\frac{\partial}{\partial\varphi}u + \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}(\cos\theta\, v) = 0$$

運動方程式:

$$\left[\frac{\partial}{\partial t} + u\frac{1}{\cos\theta}\frac{\partial}{\partial\varphi} + v\frac{\partial}{\partial\theta}\right]u - uv\tan\theta - 2\Omega\sin\theta\, v = -\frac{1}{\cos\theta}\frac{\partial p}{\partial\varphi} + f_\varphi$$

$$\left[\frac{\partial}{\partial t} + u\frac{1}{\cos\theta}\frac{\partial}{\partial\varphi} + v\frac{\partial}{\partial\theta}\right]v + u^2\tan\theta + 2\Omega\sin\theta\, u = -\frac{\partial p}{\partial\theta} + f_\theta$$

第1式は 角運動量保存.

$$\boxed{\left[\frac{\partial}{\partial t} + u\frac{1}{\cos\theta}\frac{\partial}{\partial\varphi} + v\frac{\partial}{\partial\theta}\right]\{(u+\Omega\cos\theta)\cos\theta\} = -\frac{\partial p}{\partial\varphi} + f_\varphi\cos\theta}$$

フラックス形では,

$$\frac{\partial}{\partial t}u\cos\theta + \frac{1}{\cos\theta}\frac{\partial}{\partial\varphi}u^2\cos\theta + \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\cos^2\theta\, v\,(u+\Omega\cos\theta) = -\frac{\partial p}{\partial\varphi} + f_\varphi\cos\theta$$

ロスビー波 (2次元非発散球面)

渦度方程式*

$k \times \underline{\mathrm{grad}\, \psi}$

$$\zeta \equiv (\nabla \times \boldsymbol{v})_r$$

$$= \frac{1}{\cos\theta}\frac{\partial}{\partial\varphi} v - \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}(\cos\theta\, u)$$

$$\begin{matrix} \theta & \varphi & 1 \\ \frac{\partial}{\partial\theta} & \frac{\partial}{\partial\varphi} & 1 \\ \frac{\partial\psi}{\partial\theta} & \frac{\partial\psi}{\partial\varphi} & 0 \end{matrix}$$

$$\left[\frac{\partial}{\partial t} + \frac{u}{\cos\theta}\frac{\partial}{\partial\varphi} + v\frac{\partial}{\partial\theta}\right](\zeta + 2\Omega\sin\theta) = F$$

$$F \equiv \frac{1}{\cos\theta}\frac{\partial}{\partial\varphi} f_\theta - \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}(\cos\theta\, f_\varphi)$$

流線関数：ψ

$$u = -\frac{\partial\psi}{\partial\theta}$$

$$v = \frac{1}{\cos\theta}\frac{\partial\psi}{\partial\varphi}$$

$$\zeta = \frac{1}{\cos^2\theta}\frac{\partial^2\psi}{\partial\varphi^2} + \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial\psi}{\partial\theta}\right) \equiv \nabla^2\psi$$

$$= \frac{1}{\cos\theta}\frac{\partial v}{\partial\varphi} - \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}(\cos\theta\, u)$$

$$\frac{\partial}{\partial t}(\zeta + 2\Omega\sin\theta) + \frac{1}{\cos\theta} J(\psi, \zeta + 2\Omega\sin\theta) = F$$

$$J(\psi, 8) \equiv \frac{\partial\psi}{\partial\varphi}\frac{\partial 8}{\partial\theta} - \frac{\partial\psi}{\partial\theta}\frac{\partial 8}{\partial\varphi}$$

---

*

$$\frac{\partial}{\partial t}\boldsymbol{v} + (\zeta + 2\Omega) \times \boldsymbol{v} + \nabla\frac{\boldsymbol{v}^2}{2} = -\nabla p$$

$$\downarrow \nabla\times$$

$$\frac{\partial}{\partial t}\zeta + \boldsymbol{v}\cdot\nabla(\zeta + 2\Omega) - (\zeta + 2\Omega)\cdot\nabla\boldsymbol{v} = 0$$

$$\begin{matrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & 0 \\ F_x & F_y & 0 \end{matrix}$$

ロスビー波（2次元非発散球面） 3

線形化された渦度方程式.

世界を 平均流 $\bar{u}$ （又は $\bar{\zeta} \equiv -\frac{1}{\cos\theta}\frac{\partial}{\partial\theta}(\cos\theta\,\bar{u})$ ）のまわりで線形化する.

$u = \bar{u} + u' + u^{(2)} + \cdots$

$v = v' + v^{(2)} + \cdots$

$\zeta = \bar{\zeta} + \zeta' + \zeta^{(2)} + \cdots$, $\zeta' \equiv \frac{1}{\cos\theta}\frac{\partial v'}{\partial\varphi} - \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}(\cos\theta\,u')$

$= \left[\frac{1}{\cos^2\theta}\frac{\partial^2}{\partial\varphi^2} + \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial\theta}\right)\right]\psi'$

$\psi = \bar{\psi} + \psi' + \psi^{(2)} + \cdots$

$\frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial}{\partial\theta}\right)\psi'\ \frac{\partial\psi'}{\partial\varphi}$

$\frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left[\left(\cos\theta\frac{\partial\psi'}{\partial\theta}\right)\frac{\partial\psi'}{\partial\varphi}\right]$

$-\frac{\psi'}{\cos\theta}\ \frac{\partial\psi'}{\partial\theta}\ \frac{\partial^2\psi'}{\partial\varphi\partial\theta}$

ただし

$$\text{“}-\text{”} \equiv \frac{1}{2\pi}\int_0^{2\pi} d\varphi$$

渦度方程式.

$$\boxed{\frac{\partial}{\partial t}\zeta' + \frac{1}{\cos\theta}J(\bar{\psi}, \zeta') + \frac{1}{\cos\theta}J(\psi', \bar{\zeta} + 2\Omega\sin\theta) = F'}$$

i.e.,

$$\frac{\partial}{\partial t}\zeta' + \bar{u}\frac{1}{\cos\theta}\frac{\partial}{\partial\varphi}\zeta' + \frac{1}{\cos\theta}\frac{\partial}{\partial\varphi}\psi' \cdot \bar{q}_\theta = F'$$

(2-1)

ただし $\bar{q}_\theta \equiv \frac{\partial}{\partial\theta}(\bar{\zeta} + 2\Omega\sin\theta)$

擾乱のエンストロフィー.

$$\frac{\partial}{\partial t}\frac{1}{2}\zeta'^2 + \frac{\bar{u}}{\cos\theta}\frac{\partial}{\partial\varphi}\frac{1}{2}\zeta'^2 + \frac{\bar{q}_\theta}{\cos\theta}\zeta' \cdot \frac{\partial\psi'}{\partial\varphi} = F'\zeta'$$

$\frac{\partial\zeta}{\partial t} +$

ロスビー波（2次元非発散球面）

擾乱のエンストロフィーの平均値

(2-31)

$$\frac{\partial}{\partial t}\frac{1}{2}\overline{\zeta'^2} + \frac{\bar{\zeta}_\theta}{\cos\theta}\cdot\overline{\zeta'\frac{\partial\psi'}{\partial\varphi}} = \overline{F'\zeta'}$$

$\nu\zeta'$　∇F

2次の量に関する方程式

$$\frac{\partial}{\partial t}\zeta^{(2)} + \frac{1}{\cos\theta}J(\bar{\psi}, \zeta^{(2)}) + \frac{1}{\cos\theta}J(\psi^{(2)}, \bar{\zeta} + 2\Omega\sin\theta)$$

$$+ \frac{1}{\cos\theta}J(\psi', \zeta') = F^{(2)}$$

i.e.,

$$\frac{\partial}{\partial t}\zeta^{(2)} - \frac{\bar{u}}{\cos\theta}\frac{\partial}{\partial\varphi}\zeta^{(2)} + \frac{\bar{\zeta}_\theta}{\cos\theta}\frac{\partial\psi^{(2)}}{\partial\varphi}$$

$$+ \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left(\frac{\partial\psi'}{\partial\varphi}\zeta'\right) - \frac{1}{\cos\theta}\frac{\partial}{\partial\varphi}\left(\frac{\partial\psi'}{\partial\theta}\zeta'\right) = F^{(2)}$$

2次の平均量に関する方程式

$$\boxed{\frac{\partial}{\partial t}\bar{\zeta}^{(2)} + \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left(\overline{\frac{\partial\psi'}{\partial\varphi}\zeta'}\right) = \bar{F}^{(2)}}$$

i.e.,　$\bar{\zeta}^{(2)} = -\frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left[\cos\theta\,\bar{u}^{(2)}\right]$　により

$$\boxed{\frac{\partial}{\partial t}\bar{u}^{(2)} - \frac{1}{\cos\theta}\overline{\frac{\partial\psi'}{\partial\varphi}\zeta'} = \bar{f}_\varphi^{(2)}}$$

(2-4)

$\nu\zeta'$

$\frac{1}{\rho}\nabla F = \nu\zeta'$

ロスビー波 (2次元非発散球面)

ii) WKB近似のために。

擾乱のエンストロフィーの式を フラックス形に書きなおす。

$$\zeta'\frac{\partial\psi}{\partial\varphi} = \left[\frac{1}{\cos^2\theta}\frac{\partial^2\psi'}{\partial\varphi^2} + \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial\psi'}{\partial\theta}\right)\right]\frac{\partial\psi}{\partial\varphi}$$

$$= \frac{\partial}{\partial\varphi}\frac{1}{2}\left(\frac{1}{\cos\theta}\frac{\partial\psi'}{\partial\varphi}\right)^2 + \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial\psi'}{\partial\theta}\frac{\partial\psi'}{\partial\varphi}\right) - \frac{\partial\psi'}{\partial\theta}\frac{\partial^2\psi'}{\partial\theta\partial\varphi}$$

$$= \frac{\partial}{\partial\varphi}\left[\frac{1}{2}\left(\frac{1}{\cos\theta}\frac{\partial\psi'}{\partial\varphi}\right)^2 - \frac{1}{2}\left(\frac{\partial\psi'}{\partial\theta}\right)^2\right] + \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial\psi'}{\partial\theta}\frac{\partial\psi'}{\partial\varphi}\right)$$

よって

$$\frac{\partial}{\partial t}\frac{1}{2}\zeta'^2 + \frac{\bar{u}}{\cos\theta}\frac{\partial}{\partial\varphi}\frac{1}{2}\zeta'^2$$

$$+ \frac{\bar{q}_\theta}{\cos\theta}\left\{\frac{\partial}{\partial\varphi}\left[\frac{1}{2}\left(\frac{1}{\cos\theta}\frac{\partial\psi'}{\partial\varphi}\right)^2 - \frac{1}{2}\left(\frac{\partial\psi'}{\partial\theta}\right)^2\right]\right.$$

$$\left. + \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left(\cos\theta\frac{\partial\psi'}{\partial\theta}\frac{\partial\psi'}{\partial\varphi}\right)\right\} = F'\zeta'$$

$\bar{q}_\theta$ の時間変化を "2次" の量としてムシすると、

$$\frac{\partial}{\partial t}\frac{\frac{1}{2}\zeta'^2}{\bar{q}_\theta}\cdot\cos\theta + \frac{1}{\cos\theta}\frac{\partial}{\partial\varphi}\left[\frac{1}{2}\zeta'^2\cdot\frac{\bar{u}}{\bar{q}_\theta} + \frac{1}{2}\left(\frac{1}{\cos\theta}\frac{\partial\psi'}{\partial\varphi}\right)^2 - \frac{1}{2}\left(\frac{\partial\psi'}{\partial\theta}\right)^2\right]\cdot\cos\theta$$

$$+ \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left[\cos\theta\frac{\partial\psi'}{\partial\theta}\frac{\partial\psi'}{\partial\varphi}\right] = \frac{F'\zeta'\cos\theta}{\bar{q}_\theta}$$

ロスビー波 (2次元非発散球面)

保存量は

$$A \equiv \boxed{\frac{\frac{1}{2}\zeta'^2}{q_\theta}\cos\theta}$$

$$\mathbb{F} \equiv \left(A\bar{u} + \frac{1}{2}\left(\frac{1}{\cos\theta}\frac{\partial\psi'}{\partial\varphi}\right)^2\cos\theta - \frac{1}{2}\left(\frac{\partial\psi'}{\partial\theta}\right)^2\cos\theta,\ \frac{\partial\psi'}{\partial\theta}\frac{\partial\psi'}{\partial\varphi} \right)$$

$$= \left(A\bar{u} + \frac{1}{2}(v'^2 - u'^2)\cos\theta,\ u'v'\cos\theta \right)$$

保存則は

$$\frac{\partial A}{\partial t} + \nabla\cdot\mathbb{F} = \frac{F'\zeta'\cos\theta}{q_\theta}.$$

ファクター "$\cos\theta$" に注意..

平均値については ... やはり q_θ の時間変化を 2次としてムシして

$$\frac{\partial \bar{u}^{(2)}}{\partial t} + \frac{\partial}{\partial t}\frac{\overline{\frac{1}{2}\zeta'^2}}{\bar{q}_\theta} = \overline{F_\varphi}^{(2)} + \frac{\overline{F'\zeta'}}{\bar{q}_\theta}.$$

保存量でかくためには $\cos\theta$ をかけて.

$$\frac{\partial}{\partial t}\bar{u}^{(2)}\cos\theta + \frac{\partial}{\partial t}\bar{A} = F_\varphi^{(1)}\cos\theta + \frac{\overline{F'\zeta'}}{\bar{q}_\theta}\cos\theta.$$

これをもって 擾乱は 角運動量を保存的に輸送する

と...

いう，　$\bar{A} \Leftrightarrow \bar{u}^{(2)}(\omega,t)$　をおきかえて
考えればよい．

ロスビー波（2次元非発散球面）

WKB近似

位相関数 Θ を導入する.

波数 $\mathbf{k} = (k, l)$ ， 振動数 ω

$$k \equiv \frac{1}{\cos\theta}\frac{\partial\Theta}{\partial\varphi}$$

$$l \equiv \frac{\partial\Theta}{\partial\theta}.$$

$$\omega \equiv -\frac{\partial\Theta}{\partial t}$$

線形化方程式から.

$$\zeta' = -(k^2+l^2)\psi'$$

分散関係式は.

$$\omega = \bar{u}k - \frac{\bar{q}_\theta k}{k^2+l^2} = \frac{\bar{u}}{\cos\theta}(k\cos\theta) - \frac{\frac{\bar{q}_\theta}{\cos\theta}(k\cos\theta)}{\frac{(k\cos\theta)^2}{\cos^2\theta} + l^2}$$

群速度

$$C_{g\varphi} = \bar{u} + \frac{\bar{q}_\theta(k^2-l^2)}{(k^2+l^2)^2} = \bar{u} + (k^2-l^2)\frac{\hat{\omega}^2}{\bar{q}_\theta k^2}$$

$$C_{g\theta} = \frac{\bar{q}_\theta\, 2kl}{(k^2+l^2)^2} = \frac{2l}{k}\cdot\frac{\hat{\omega}^2}{\bar{q}_\theta}$$

ただし

$$C_{g\varphi} \equiv \frac{\partial\omega}{\partial(k\cos\theta)}\cdot\cos\theta$$

$$C_{g\theta} \equiv \frac{\partial\omega}{\partial l}$$

ロスビー波（2次元非発散球面）

波数保存則

$$\frac{\partial k\cos\theta}{\partial t} = -\frac{\partial \omega}{\partial \varphi} \quad , \quad \frac{\partial l}{\partial t} = -\frac{\partial \omega}{\partial \theta}$$

$$\frac{\partial k\cos\theta}{\partial \theta} = \frac{\partial l}{\partial \varphi}$$

よって

$$\frac{\partial}{\partial t}(k\cos\theta) = -\frac{\partial \omega}{\partial \varphi}$$

$$= -\frac{\partial \omega}{\partial (k\cos\theta)}\frac{\partial (k\cos\theta)}{\partial \varphi} - \frac{\partial \omega}{\partial l}\frac{\partial l}{\partial \varphi} - \frac{\partial \omega}{\partial \varphi}$$

$$= -\underset{\cos\phi}{C_{g\varphi}}\frac{\partial k\cos\phi}{\partial \varphi} - C_{g\theta}\frac{\partial (k\cos\theta)}{\partial \theta} - \frac{\partial \omega}{\partial \varphi}$$

$$\frac{\partial}{\partial t} l = -\frac{\partial \omega}{\partial \theta}$$

$$= -\frac{\partial \omega}{\partial (k\cos\theta)}\frac{\partial (k\cos\theta)}{\partial \theta} - \frac{\partial \omega}{\partial l}\frac{\partial l}{\partial \theta} - \frac{\partial \omega}{\partial \theta}$$

$$= -\underset{\cos\phi}{C_{g\varphi}}\frac{\partial l}{\partial \varphi} - C_{g\theta}\frac{\partial l}{\partial \theta} - \frac{\partial \omega}{\partial \theta}$$

$$\frac{\partial \omega}{\partial t} = +\frac{\partial \omega}{\partial (k\cos\theta)}\frac{\partial (k\cos\theta)}{\partial t} + \frac{\partial \omega}{\partial l}\frac{\partial l}{\partial t} + \frac{\partial \omega}{\partial t}$$

$$= -\underset{\cos\phi}{C_{g\varphi}}\frac{\partial \omega}{\partial \varphi} - C_{g\theta}\frac{\partial \omega}{\partial \theta} + \frac{\partial \omega}{\partial t}$$

まとめ

$$D_t(k\cos\theta) = -\frac{\partial \omega}{\partial \varphi}$$

$$D_t l = -\frac{\partial \omega}{\partial \theta}$$

$$D_t \omega = \frac{\partial \omega}{\partial t}$$

ただし $D_t \equiv \partial_t + C_{g\varphi}\underset{\frac{1}{\cos\phi}}{\partial_\varphi} + C_{g\theta}\partial_\theta$

$\omega = \omega(k\cos\theta, l; \theta, \varphi, t)$

ロスビー波（2次元非発散球面） 9

基本場はθのみの関数なので、

$$D_t (k\cos\theta) = 0$$

$$D_t l = -\frac{\partial\omega}{\partial\theta}$$

$$D_t \omega = 0$$

いわゆる"波"存在領域は、 ω, k に対して、$l^2 > 0$ である条件。として与えられる。（任意性をおさえるために $k > 0$ とおう）

$$l^2 = -\frac{\bar{q}_\theta k}{\omega - \bar{u}k} - k^2 > 0$$

i.e.,

$$-\frac{\bar{q}_\theta}{\omega - \bar{u}k} > k$$

波線にそって $k\cos\theta \equiv k_M$ (const) であることを使って書きかえると

$$-\frac{2\Omega_M \cdot \cos^2\theta}{\omega - \frac{\bar{u}}{\cos\theta}\cdot k_M} > k_M$$

ただし . . .

$$2\Omega_M \equiv \frac{\bar{q}_\theta}{\cos\theta} = \frac{1}{\cos\theta}\partial_\theta \bar{\zeta} + 2\Omega$$

$$= -\frac{1}{\cos\theta}\partial_\theta\left[\frac{1}{\cos\theta}\partial_\theta(\cos\theta\,\bar{u})\right] + 2\Omega$$

$\theta \to \pm\frac{\pi}{2}$ においては、

$$\Omega_M < \infty \ , \quad \frac{\bar{u}}{\cos\theta} < \infty$$

なので 不等式の |左辺| → 0. したがって いわゆる 転移緯度 がどこかに 存在する.

ロスビー波 (2次元非発散球面)

2次の量 の伝播.

WKB すれば,

$$\frac{1}{2}\psi'^2\cos\theta = \frac{1}{2}\frac{1}{(k^2+l^2)^2}\zeta'^2\cos\theta = \frac{A\bar{q}_\theta}{(k^2+l^2)^2}$$

に注意して

$$\frac{\partial}{\partial t}A + \frac{1}{\cos\theta}\frac{\partial}{\partial\varphi}\left[\left(\bar{u} + \frac{k^2-l^2}{(k^2+l^2)^2}\bar{q}_\theta\right)A\right]$$
$$+ \frac{1}{\cos\theta}\frac{\partial}{\partial\theta}\left[\cos\theta\cdot\frac{2kl}{(k^2+l^2)^2}\bar{q}_\theta A\right] = \frac{F'\zeta'\cos\theta}{\bar{q}_\theta}$$

すなわち

$$\frac{\partial}{\partial t}A + \nabla\cdot(c_g A) = \frac{F'\zeta'\cos\theta}{\bar{q}_\theta}$$

平均すれば

$$\frac{\partial}{\partial t}\bar{A} + \frac{1}{\cos\theta}\partial_\theta(\cos\theta\ c_{g\theta}\bar{A}) = \frac{\overline{F'\zeta'}\cos\theta}{\bar{q}_\theta}$$

ロスビー波（2次元非発散球面） 11

例題. Rayleigh damping と 強制問題. （ある緯度で波を出しつづけて..）

$$\cancel{\frac{\partial}{\partial t}\bar{A}} + \frac{1}{\cos\theta}\partial_\theta\left(\cos\theta\, c_{g\theta}\bar{A}\right) = -\frac{2\bar{A}}{\tau}$$

↳ 定常になってはる.

$$\bar{A}\cos\theta \propto e^{-2\int\frac{d\theta}{\tau c_{g\theta}}}$$

$$\frac{\partial}{\partial t}\bar{u}\cos\theta = \bar{F}_\varphi\cos\theta - \frac{2\bar{A}}{\tau}$$

$$= \bar{F}_\varphi\cos\theta - \underline{\underline{\frac{2A_0\, e^{-2\int\frac{d\theta}{\tau c_{g\theta}}}}{\tau\cos\theta}}}$$

このような東向加速が生じる.

```
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#1
 DOC NO         2059985
 TITLE          Large-scale two-dimensional turbulence in the atmosphere
 AUTHOR         Boer, G.J.
                Shepherd, T.G.
 AUTHOR AT      Canadian Climate Centre, Downsview, Ontario, Canada
 TAKEN FROM     J. Atmos. Sci. (USA)
 CODEN          JAHSAK
 VOL            vol.40, no.1
 PAGE           164-84
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#2
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 TITLE          Mean motions induced by baroclinic instability in a jet
 AUTHOR         Shepherd, T.G.
 AUTHOR AT      Dept. of Meteorology & Phys. Oceanography, MIT, Cambridge, MA,
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 CODEN          GAFDD3
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 TITLE          Time development of small disturbances to plane Couette flow
 AUTHOR         Shepherd, T.G.
 AUTHOR AT      Dept. of Appl. Math. & Theor. Phys., Cambridge Univ., England
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 TITLE          A spectral view of nonlinear fluxes and stationary-transient
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 AUTHOR         Shepherd, T.G.
 AUTHOR AT      Center for Meteorol. & Phys. Oceanogr., MIT, Cambridge, MA, USA
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 TITLE          Inhomogeneous two-dimensional turbulence in the atmosphere
 AUTHOR         Shepherd, T.G.
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 DATE OF CONF   1-4 July 1986

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             zonal jet
 AUTHOR      Shepherd, T.G.
 AUTHOR AT   Center for Meteorol. & Phys. Oceanogr., MIT, Cambridge, MA, USA
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 CODEN       JFLSA7
 VOL         vol.183
 PAGE        467-509
 PUBL DATE   Oct. 1987

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 TITLE       An exact local conservation theorem for finite-amplitude
             disturbances to non-parallel shear flows, with remarks on
             Hamiltonian structure and on Arnol'd's stability theorems
 AUTHOR      McIntyre, M.E.
             Shepherd, T.G.
 AUTHOR AT   Dept. of Appl. Math. & Theor. Phys., Cambridge Univ., UK
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 VOL         vol.181
 PAGE        527-65
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 DOC NO      3063739
 TITLE       Non-ergodicity of inviscid two-dimensional flow on a beta-plane
             and on the surface of a rotating sphere
 AUTHOR      Shepherd, T.G.
 AUTHOR AT   Dept. of Appl. Math., Cambridge Univ., UK
 TAKEN FROM  J. Fluid Mech. (UK)
 CODEN       JFLSA7
 VOL         vol.184
 PAGE        289-302
 PUBL DATE   Nov. 1987

#9
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 TITLE       On Rossby waves modified by weak sinusoidal shear
 AUTHOR      Shepherd, T.G.
 AUTHOR AT   Dept. of Appl. Math. & Theor. Phys., Cambridge Univ., UK
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 PAGE        225-37
 PUBL DATE   1988

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             model
 AUTHOR      Shepherd, T.G.
 AUTHOR AT   Dept. of Appl. Math. & Theor. Phys., Cambridge Univ., UK
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 AUTHOR      Shepherd, T.G.
 AUTHOR AT   Dept. of Appl. Math. & Theor. Phys., Cambridge Univ., UK
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 DOC NO         3358230
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                for the instability of steady equivalent-barotropic flows
 AUTHOR         Shepherd, T.G.
 AUTHOR AT      Dept. of Phys., Toronto Univ., Ont., Canada
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 CODEN          NIFCAS
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 TITLE          Nonlinear saturation of baroclinic instability. II. Continuously
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 AUTHOR         Shepherd, T.G.
 AUTHOR AT      Dept. of Appl. Math. & Theor. Phys., Cambridge Univ., UK
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 VOL            vol.46, no.7
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 DOC NO         3554940
 TITLE          The importance of surface pressure changes in the response of
                the atmosphere to zonally-symmetric thermal and mechanical
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 AUTHOR         Haynes, P.H.
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 TITLE          A general method for finding extremal states of Hamiltonian
                dynamical systems, with applications to perfect fluids
 AUTHOR         Shepherd, T.G.
 AUTHOR AT      Dept. of Phys., Toronto Univ., Ont., Canada
 TAKEN FROM     J. Fluid Mech. (UK)
 CODEN          JFLSA7
 VOL            vol.213
 PAGE           573-87
 PUBL DATE      April 1990

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 DOC NO         3754136
 TITLE          Isovortical constraints on the statistical-dynamical behaviour
                of strongly nonlinear two-dimensional and quasi-geostrophic flow
 AUTHOR         Shepherd, T.G.
 AUTHOR AT      Dept. of Phys., Toronto Univ., Ont., Canada
 TAKEN FROM     Topological Fluid Mechanics. Proceedings of the IUTAM Symposium
 NO OF PAGE     xviii+805
 PAGE           278-87
 PUBLISHED BY   Cambridge University Press
 PUBL DATE      1990
 CONF AT        Cambridge, UK
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 DATE OF CONF  13-18 Aug. 1989

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 TITLE         A natural method for finding stable states of Hamiltonian
               systems
 AUTHOR        Vallis, G.K.
               Carnevale, G.F.
               Shepherd, T.G.
 AUTHOR AT     Div. of Natural Sci., California Univ., Santa Cruz, CA, USA
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 PUBL DATE     1990
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 TITLE         On the interpretation of Andrews' theorem (nonlinear flows)
 AUTHOR        Carnevale, G.F.
               Shepherd, T.G.
 AUTHOR AT     Scripps Instn. of Oceanogr., California Univ., San Diego, La
               Jolla, CA, USA
 TAKEN FROM    Geophys. Astrophys. Fluid Dyn. (UK)
 CODEN         GAFDD3
 VOL           vol.51, no.1-4
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 DOC NO        3912298
 TITLE         On the 'downward control' of extratropical diabatic circulations
               by eddy-induced mean zonal forces
 AUTHOR        Haynes, P.H.
               McIntyre, M.E.
               Shepherd, T.G.
               Shine, K.P.
 AUTHOR AT     Dept. of Appl. Math. & Theor. Phys., Cambridge Univ., UK
 TAKEN FROM    J. Atmos. Sci. (USA)
 CODEN         JAHSAK
 VOL           vol.48, no.4
 PAGE          651-78
 PUBL DATE     15 Feb. 1991

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 DOC NO        3953163
 TITLE         Nonlinear stability and the saturation of instabilities to
               axisymmetric vortices
 AUTHOR        Shepherd, T.G.
 AUTHOR AT     Dept. of Phys., Toronto Univ., Ont., Canada
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 CODEN         EJBFEV
 VOL           vol.10, no.2, suppl.
 PAGE          93-8
 PUBL DATE     1991
 CONFERENCE    Nonlinear Hydrodynamic Stability and Transition. IUTAM Symposium
 CONF AT       Nice, France
 DATE OF CONF  3-7 Sept. 1990

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 DOC NO        4018125
 TITLE         The stability of a two-dimensional vorticity filament under
               uniform strain
 AUTHOR        Dritschel, D.G.
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	Haynes, P.H. Juckes, M.N. Shepherd, T.G.
AUTHOR AT	Dept. of Appl. Math. & Theor. Phys., Cambridge Univ., UK
TAKEN FROM	J. Fluid Mech. (UK)
CODEN	JFLSA7
VOL	vol.230
PAGE	647-65
PUBL DATE	Sept. 1991

5/18

Bouille

$$\frac{N}{f} \propto \left(\frac{L_z}{L_x}\right)^{-1}$$

$L_x \sim 1000$ k

T_{xz} - L_z - 10 km

$\Delta z \sim$ 2-5 km

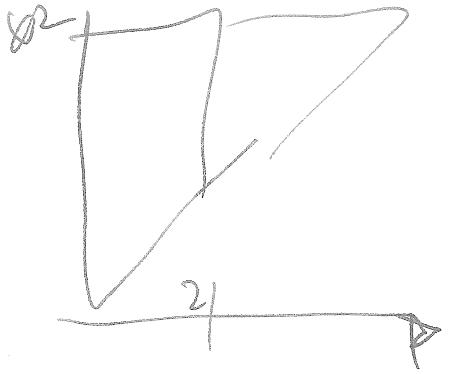
$\frac{1}{4}$

$\boxed{\sim 500 \text{ m}}$

$\Delta z \sim 0.5$ km

ℓt 51 × 8

Δt



thermal damping

- vertical diffusion

$$\frac{\partial u}{\partial t} = \; + \frac{1}{\rho}\frac{\partial}{\partial z}\left(\rho \underline{K} \frac{\partial u}{\partial z}\right)$$

$K = f(Ri)\, K_N$

$K_N = \ell^2 \left|\frac{\partial \bar{V}}{\partial z}\right|$ m²/s

$f = 1 \quad Ri = 0$

$f \to 0 \quad Ri \to 4$

$\frac{\partial u}{\partial t} = \textcircled{L} u$

$L(u)$

time-split implicit method

$$\frac{u^{n+1}-u^{n-1}}{\Delta t} = L^{n-1} u^{n-1}$$

$L_m \sim 40$

$l_H \sim$)

$l = 30$ free atmosp lane

- Fels et al. JAS 37 1980

  SKIHI

- Andrews, et al. JAS 40 1983

To~~p~~ top

4 level.

90km

Rayleigh friction

$\frac{\partial V^*}{\partial t} \simeq -\alpha V^*$ wave amp

5/19 OW PostDoc.

Joan Alexander
alexand@atmos.washington.edu

Sata . Yamada.
<u>wavelet analysis</u>.

IVE

Holton (78?)
Salby & Garcia
Webster e

generation of equatorial waves

Zhang Chidong.

4/20 •
Jim's grad stu
• Karen.

CCM2
downward control

. UARS
H_2O
CH_4

. Obs.
UKMO

SCUOLA INTERNAZIONALE DI FISICA

CIX ~ 93

"Aspects of Nonlinear Physics of the Ocean"

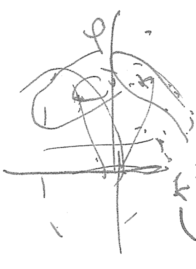
Italian Society of Physics.

Bo.

T. Shepherd : Ph.D thesis ~

stochastic forcing

$E(k, \varphi)$



$\overline{\sigma}$ $\frac{d\overline{\sigma}}{d\mu}$ JFM '82 ~ 4

(Vallis) 91 ~ 2.