4. TURBULENT PLUMES

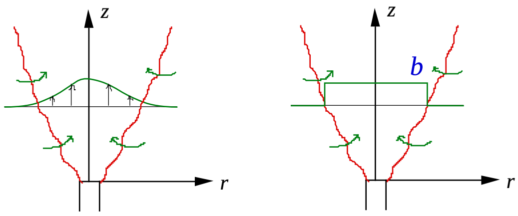
4.1

1. Preliminaries

- Examples of cigarettes, smoke stacks, volcanic vents.
- Convection in the form of:
  - (1) buoyant plumes
  - (2) momentum jets
  - (3) finite volume thermals
- A rising light fluid is dynamically equivalent to a falling heavy fluid in B.a.
- Unsteady at Re = O(10), fully turbulent at Re ~O(10^3).
- High Re motions are independent of ν (and of κ) and hence of Re.
- Assume for simplicity, a calm ambient (background).
- There is a sharp boundary (between plume and ambient) at any point in time, with strong temporal variation.
- ***Entrainment*** of ambient by eddies leads to increasing width of the plume. Mixing takes place within plume in a manner independent of ν, κ etc.
- ***U***, ρ_o' are intermittent, but consider temporal averages over a short time to lead to averages independent of short time, though there may still be long-time variations.

4.2

- All quantities vary as a gaussian in the horizontal, but often a uniform approximation is made resulting in a top-hat profile.



2. Quantification of a plume

at source: U, A, $g' = g(\rho_a - \rho)/\rho$

reduced gravity

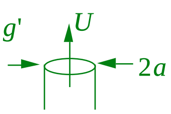
vertical velocity

(i) Specific mass flux $\quad Q = \int w\,dA \sim \pi a^2 U \equiv L^3 T^{-1}$

per unit mass

(ii) Specific momentum flux $\quad M = \int w^2\,dA \sim \pi a^2 U^2 \equiv L^4 T^{-2}$

(iii) Specific buoyancy flux $\quad F = \int wg'\,dA \; = 2\pi \int wg' r\,dr \sim g'\pi a^2 U \equiv L^4 T^{-3}$



$$U = M/Q, \quad A = Q^2/M, \quad g' = F/Q$$

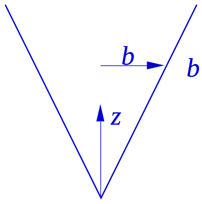
4.3

F, Q, M are functions of z with initial values (at source) of F_o Q_o and M_o

Consider a pure buoyancy source: $F_o \neq 0, \quad M_o = Q_o = 0$ (eg. a cigarette)

Assume it is discharging into a uniform ambient. Then F remains constant.

Far from the (small) source there is no input lengthscale and so



b radius b z

Dimensional analysis and experiments yield,

$$w = 4.7\, F_o^{1/3} z^{-1/3} \exp\left(-96\, r^2/z^2\right)$$

Experiment

$$g' \equiv \frac{g(\rho_a - \rho)}{\rho_a}$$

ρ_a = Ambient density

$$= 11\, F_o^{2/3} z^{-5/3} \exp\left(-71\, r^2/z^2\right)$$

$b = 0.17\, z$ Q wb^2 $F_o^{1/3} z^{5/3}$ M $w^2 b^2$ $F_o^{2/3} z^{4/3}$

increases due to entrainment increases due to working of buoyancy forces

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4.4

3. Morton, Turner and Taylor (1956) plumes

Turner 1969 & 1979

Linden 2000

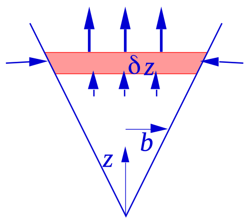
b z $u = \alpha w$

entrainment constant

inflow velocity vertical velocity

With this assumption, can look in general at equations for plume motion. These can be obtained from first principles or by boundary-layer equations (Appendix).

1) Conservation of mass $\overline{w}^{\gamma} = b^{-\gamma} \int w^{\gamma}\, r\, dr \quad (\gamma = 1, 2)$



Consider a strip of height δz

$$\delta\left(\pi b^2 \overline{w}\right) = 2\pi b \delta z u \qquad w \qquad = 2\pi\alpha b \overline{w} \delta z$$

$$\frac{d}{dz}\left(b^2 \overline{w}\right) = 2\alpha\, b\, \overline{w} \qquad \text{(I)}$$

2) Conservation of momentum

$$\delta\left[\pi b^2 \rho \overline{w}^2\right] = \pi b^2 \delta z g(\rho - \rho_a) \quad \text{i.e.} \quad \frac{d}{dz}\left(b^2 \overline{w}^2\right) = b^2 g\left(\frac{\rho - \rho_a}{\rho}\right)$$

In the Boussinesq approximation, can say $\rho \approx \rho_a$

So $$\frac{d}{dz}\left(b^2 \overline{w}^2\right) = b^2 g' \qquad \text{(II)}$$

4.5

3) Conservation of buoyancy

(Consider ρ as some reference density)

$$\delta\left[\pi b^2(\rho \; - \rho)\overline{w}\right] = 2\pi b \delta z u(\rho \; - \rho)$$

$$\frac{d}{dz}\left[b^2(\rho \; - \rho)\overline{w}\right] = 2\alpha b\overline{w}(\rho \; - \rho_a)$$

$$= (\rho \; - \rho_a)\frac{d}{dz}\left(b^2\overline{w}\right) \qquad \text{from (I)}$$

$$= \frac{d}{dz}\left[b^2\overline{w}(\rho \; - \rho_a)\right] + b^2\overline{w}\frac{d\rho_a}{dz}$$

$$\frac{d}{dz}\left[b^2 g'\overline{w}\right] = b^2\overline{w}\frac{g}{\rho}\frac{d\rho_a}{dz} = -N^2(z)b^2\overline{w} \qquad \text{(III)}$$

Solutions exist for a variety of different cases.

4.6

Alternatively, in terms of Q, M and F

$$\frac{dQ}{dz} = 2\pi^{3/2}\alpha M^{1/2}$$

$$\frac{dM}{dz} = FQ/M$$

$$\frac{dF}{dz} = -N^2 Q$$

$$Q = Q_0, \quad M = M_0, \quad F = F_0 \quad (z = 0).$$

4.7

Case (a) $N^2 = -\frac{g}{\rho_o}\frac{d\rho}{dz} \equiv 0$ (i.e. Uniform Ambient)

From (III), F remains constant.

If further $Q_0 = M_0 = 0$

$$b = \frac{5}{6}\alpha z \qquad \overline{w} = \frac{5}{6\alpha}\left(\frac{9}{10}\alpha F\right)^{1/3} z^{-1/3}$$

$$g' = \frac{5F}{6\alpha}\left(\frac{9}{10}\alpha F\right)^{-1/3} z^{-5/3}$$

Note: from (I) $\frac{1}{Q}\frac{dQ}{dz} = \frac{2\alpha}{b}$

from (I) and (II) $\frac{db}{dz} = 2\alpha - \frac{1}{2}\frac{bg'}{\overline{w}^2} \begin{cases} = 2\alpha & g' = 0 \text{ jet} \\ = \frac{5\alpha}{6} & N \equiv 0 \text{ plume} \end{cases}$

$\alpha \approx 0.1$ (0.085 better fit)

4.8

Case (b) $M_0, F_0 \neq 0$ but $Q_0 = N = 0$

Define a lengthscale based on

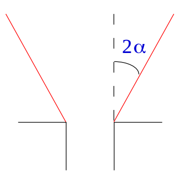
$$l_{MF} = M_0^{3/4} / F_0^{1/2}$$

Solutions in terms of $\eta = z / l_{MF}$

$\eta \to 0$ represents either $z \to 0$ with l_{MF} fixed or z fixed and $l_{MF} \to \infty$,

i.e. $F_0 \to 0$, a momentum-driven jet.

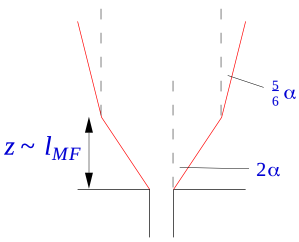
i.e. Close to the source, output acts like a momentum jet.



$\eta \to \infty$ represents either $z \to \infty$ for fixed l_{MF} or z fixed, $M_0 \to 0$, a pure buoyancy-driven plume.

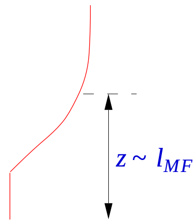
i.e. Far from the source, the plume acts like pure buoyancy. i.e. $l >> l_{MF}$

4.9

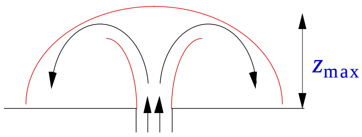


or more

accurately



Alternatively, heavy fluid can be shot out of a source



z_{max} l_{MF} In fact $z_{max} = 1.85\ l_{MF}$

4.10

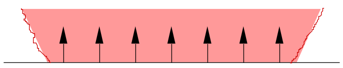
Case (c) $Q_0,\ M_0 \neq 0$ $F_0 = 0$ $N = 0$

$$l_{QM} = \frac{Q_0}{M_0^{1/2}}$$

when $z << l_{QM}$, mass flux is important

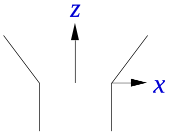
when $z >> l_{QM}$, momentum flux is important

Case (d) 2D Plumes



line source

B = 2D Buoyancy Flux



$$B = \int_0^\infty w g' \, dx \equiv \left[L^3 T^{-3} \right]$$

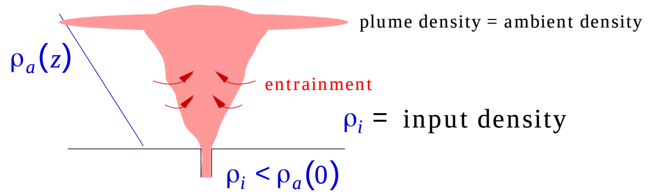
If only B_o is non-zero

w $B^{1/3}$ $B = B_0$ always

independent of z in 2D case

Case (e) $F_0,\ N \neq 0 \quad M_0 = Q_0 = 0$ 4.11

Ambient density gradient



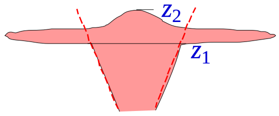
Ambient density gets less and less. Eventually the density of the plume will reach the density of the ambient.

Length scale based on F_0, N $\quad l_{FN} = F_0^{1/4} N^{-3/4}$ for N constant

Solutions show

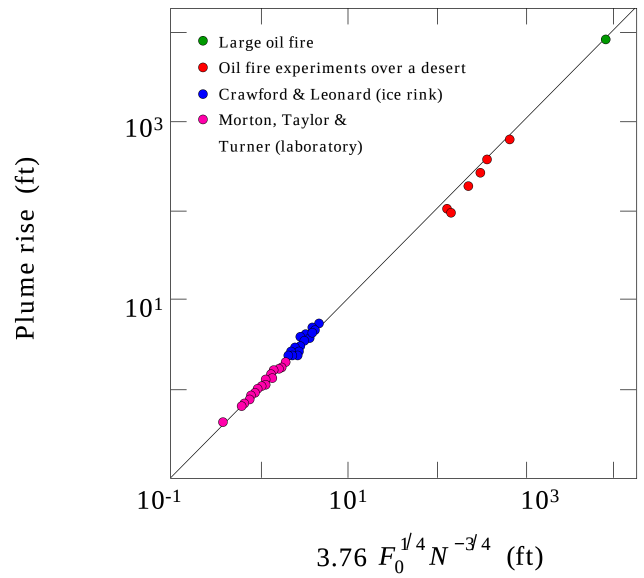
at $z = z_1 = 1.04\alpha^{-1/2} l_{FN}$ $\quad g' = 0$ (the plume runs out of buoyancy, but still has momentum)

at $z = z_2 = 1.37\alpha^{-1/2} l_{FN}$ $\quad w = 0$ (momentum gone)



where, in the calculation of z_2, it is assumed that it continues to propagate as a regular plume. By experiment, at $z = z_3 = 3.76\, l_{FN}$ the plume intrudes.

4.12



Measurements of plume rise in calm stratified surroundings compared with the theoretical relationship (in *Turner*, page199).

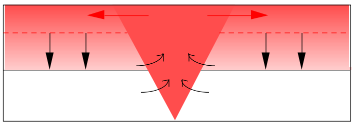
4.13

Case (f) Morton integrated the equations for

$M_0, F_0, N \neq 0 \quad Q_0 = 0$ i.e. a forced plume in a stratified ambient.

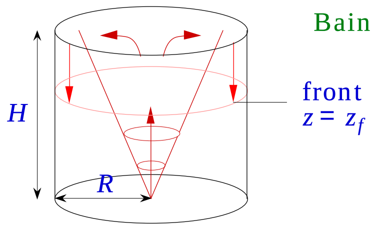
As $M_0 \uparrow$, $z_{max} \downarrow$ because near source jet entrains more fluid 2α vis a vis $\frac{6}{5}\alpha$ so, in some range of parameters, the plume does not go as high.

Case (g) Finite Box



Filling Box - influences the whole ambient

Axisymmetric plume in a finite axisymmetric box.



Baines & Turner 1969

Equations I -- III for the plume with $N^2(z)$ to be determined along with vertical velocity U behind the front.

4.14

Continuity $\quad \frac{\partial \rho_a}{\partial t} + \frac{U \partial \rho_a}{\partial z} = 0$

Density difference at front equals density difference in plume.

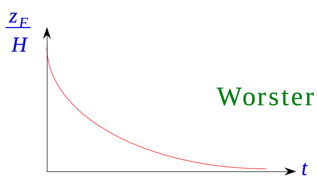
Solution

g' at $z = z_F^+$ equals $g'(z = H)$ in original plume.

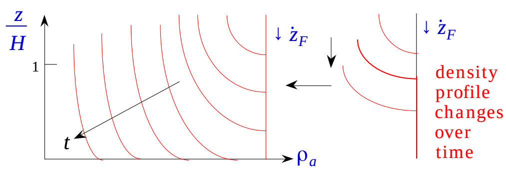
Result

$$t = \left(\frac{5}{4\alpha}\right)\left(\frac{10}{9\alpha}\right)^{1/3} R^2 F_0^{-1/3}\left(z_F^{-1/3} - H^{-2/3}\right)$$

which gives $z_F(t)$



Worster & H^2 1983



4.15

Summary of Plume Theory

- Turbulent plumes propagate and spread under action of entrainment.
- Plumes described by three input parameters, Q, M, F.
- Motion predicted by three equations for $\overline{w}$, g' and b as functions of z . Solutions of governing equations often obtainable by dimensional analysis.
- Results applicable to a very wide range of sizes.
- Filling box effects important in confined regions such as rooms, buildings, craters.

4.16

Lecture 4. Turbulent Plumes

Baines, W.D. and Turner, J.S., 1969 Turbulent convection from a source in a confined region, *J. Fluid Mech.*, **37**, 51-80.

Morton, B.R., Taylor, G.I. and Turner, J.S., 1956 Turbulent gravitational convection from maintained and instantaneous sources, *Proc. Roy. Soc.* **A 234**, 1-23.

Linden, P.F., 2000 Convection in the environment. In: *Perspectives in Fluid Dynamics: A Collective Introduction to Current Research*. G.K. Batchelor, H.K. Moffat and M.G. Worster (eds.) Cambridge University Press, pp.289-345.

Turner, J.S., 1969 Buoyant Plumes and Thermals. *Ann. Rev. Fluid Mech.*, **1**, 29-44.

Turner, J.S., 1979 *Buoyancy Effects in Fluids*. Cambridge University Press.

Worster, M.G. and Huppert, H.E., 1983 Time dependent density profiles in a filling box, *J. Fluid Mech.*, **132**, 457-466.

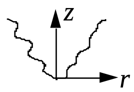
Appendix

4.A1

Equations of motion

Axisymmetric, steady flow with no swirl $\mathbf{u} = (u, 0, w)$

Ignore viscosity and diffusion



$$u\frac{\partial u}{\partial r} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial r} \qquad (1)$$

$$u\frac{\partial w}{\partial r} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho_0}\frac{\partial p}{\partial z} - \frac{g\rho}{\rho_0} \qquad (2)$$

$$u\frac{\partial p}{\partial r} + w\frac{\partial p}{\partial z} = 0 \qquad (3) \qquad\qquad \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{\partial w}{\partial z} = 0 \qquad (4)$$

The entrainment assumption

Integrate (11) across the plume

$$\int_0^\infty r\frac{\partial w}{\partial z}\,\mathrm{d}r = -\int_0^\infty \frac{\partial}{\partial r}(ru)\,\mathrm{d}r$$

$$\frac{d}{dz}\int_0^\infty rw\,\mathrm{d}r = -[ru]_0^\infty = -ru\big|_\infty \qquad (5)$$

rate of increase of volume flow in the plume is compensated by an inflow at infinity - entrainment.

4.A2

The rising plume acts like a line sink to the exterior flow.

From (5) we see that similarity theory

$$\frac{dQ}{dz} \sim z^{2/3}$$

$$\sim ru\big|_\infty \qquad \text{by (12)}$$

$$\sim ru\big|_{\text{edge of plume}}$$

Since $r \sim z$ (see (4)) $\quad u \sim z^{-1/3}$

From (2) note that $w \sim z^{-1/3}$ and so the inflow velocity at the edge of the plume has the same vertical dependence as the vertical velocity in the plume.

Assume that inflow velocity is a constant fraction α of the upward velocity in the plume - entrainment assumption (Morton, Tayor & Turner, 1956 Proc. Roy Soc. A **234**, 1-23)

In unstratified fluids the entrainment assumption and similarity theory are equivalent.

4.A3

The buoyancy flux B

$$B = 2\pi\int_0^\infty rw\frac{\rho-\rho_e}{\rho_0}\,\mathrm{d}r$$

Now

$$\frac{d}{dz}\int_0^\infty rw(\rho-\rho_e)\,\mathrm{d}r = \int_0^\infty r(\rho-\rho_e)\frac{\partial w}{\partial z}\,\mathrm{d}r \quad + \int_0^\infty rw\frac{\partial}{\partial z}(\rho-\rho_e)\,\mathrm{d}r$$

$$= -\int_0^\infty(\rho-\rho_e)\frac{\partial}{\partial r}(ru)\,\mathrm{d}r - \int_0^\infty rw\frac{\partial \rho_e}{\partial z}\,\mathrm{d}r \quad - \int_0^\infty ru\frac{\partial}{\partial r}(\rho-\rho_e)\,\mathrm{d}r \qquad \text{from (3), (4)}$$

$$= -\int_0^\infty\frac{\partial}{\partial r}\left[ru(\rho-\rho_e)\right]\mathrm{d}r - \frac{\partial \rho_e}{\partial z}\int_0^\infty rw\,\mathrm{d}r$$

$$= \left[ru(\rho-\rho_e)\right]_0^\infty - \frac{\partial \rho_e}{\partial z}\int_0^\infty rw\,\mathrm{d}r$$

$$= -\frac{\partial \rho_e}{\partial z}\int_0^\infty rw\,\mathrm{d}r \qquad (15)$$

Hence, B conserved in unstratified surroundings (where $\frac{\partial \rho_e}{\partial z} = 0$).

4.A4

Outside plume $\rho = \rho_e(z)$, $w = 0$ and (9) $\qquad \frac{\partial p}{\partial z} = -g\rho_e \qquad (13)$

If we assume the plume is thin $\left(\frac{\partial}{\partial r} >> \frac{\partial}{\partial z}\right)$, (11) $\qquad \frac{u}{w} \sim \frac{b}{z} \sim \beta << 1.$

Comparison of (8) and (9) shows that

$$\frac{\partial p}{\partial r}\bigg/\frac{\partial p}{\partial z} \sim \beta$$

and so, to first order $p = p(z)$ and (13) holds across the plume.

Now integrate (9) across the plume and use (11) & (13) →

$$\frac{d}{dz}\int_0^\infty rw^2\,dr = -\int_0^\infty rg\frac{(\rho-\rho_e)}{\rho_0}\,dr \qquad (14)$$

vertical rate of increase in momentum flux equal to work done by buoyancy force.

4.A5

Top-hat profiles

Since the flow in the plume is turbulent we can write all quantities as the sum of a mean and a fluctuating part

$$\text{eg} \quad u = \overline{u} + u', \quad \text{where} \quad \overline{u'} = 0 \qquad (16)$$

In a steady flow it is convenient to take time averages

Consider the mass conservation equation (10) and substitute

$$(\overline{u} + u')\frac{\partial(\overline{\rho} + \rho')}{\partial r} + (\overline{w} + w')\frac{\partial(\overline{\rho} + \rho')}{\partial z} = 0$$

Take average and use (16)

$$\underbrace{\overline{u}\frac{\partial\overline{\rho}}{\partial r} + \overline{w}\frac{\partial\overline{\rho}}{\partial z}}_{\text{mean fluxes}} = \underbrace{-\overline{u'\frac{\partial\rho'}{\partial r}} - \overline{w'\frac{\partial\rho'}{\partial z}}}_{\text{turbulent fluxes}}$$

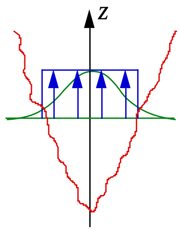
In classical plume theory it is assumed that the turbulent fluxes are negligible compared with the mean fluxes and than we have (8) - (11) rewritten with mean values eg $\overline{u}\frac{\partial\overline{\rho}}{\partial r} + \overline{w}\frac{\partial\overline{\rho}}{\partial z} = 0$

4.A6

The observed mean properties are Gaussian but we can define equivalent top-hat profiles

$$\overline{w}\,b^2 = \int_0^\infty w(r)r \;\mathrm{dr}$$

$$\overline{w}^2 b^2 = \int_0^\infty w^2(r)r \;\mathrm{dr}$$



Define the top- hat velocity $\overline{w}$ and width b

Outside the plume $\overline{w} = 0$.

Conservation equations are - with the entrainment assumption

mass flux $$\frac{d}{dz}\left(\overline{w}\,b^2\right) = 2\alpha b\overline{w} \qquad \text{(I)}$$

momentum flux $$\frac{d}{dz}\left(b^2\overline{w}^2\right) = b^2 g' \qquad \text{(II)}$$

buoyancy flux $$\frac{d}{dz}\left(b^2\overline{w}\,g'\right) = -b^2\overline{w}\,N^2(z) \qquad \text{(III)}$$