演習問題 5.6 解答例

(1)

$$
\begin{aligned}
(\nabla\psi)_{u_j} &= \frac{1}{h_j}\frac{\partial\psi}{\partial u_j}, \\
\nabla\cdot\boldsymbol{A} &= \frac{1}{h_1h_2h_3}\left(\frac{\partial A_1h_2h_3}{\partial u_1}+\frac{\partial A_2h_3h_1}{\partial u_2}+\frac{\partial A_3h_1h_2}{\partial u_3}\right)
\end{aligned}
$$

であることより，

$$
\begin{aligned}
\nabla^2\psi &= \nabla\cdot(\nabla\psi) \\
&= \frac{1}{h_1h_2h_3}\left\{\frac{\partial}{\partial u_1}\left(\frac{h_2h_3}{h_1}\frac{\partial\psi}{\partial u_1}\right)+\frac{\partial}{\partial u_2}\left(\frac{h_3h_1}{h_2}\frac{\partial\psi}{\partial u_2}\right)+\frac{\partial}{\partial u_3}\left(\frac{h_1h_2}{h_3}\frac{\partial\psi}{\partial u_3}\right)\right\}.
\end{aligned}
$$

(2)

円筒座標におけるラプラシアンは

$$
\begin{aligned}
\nabla^2\psi &= \frac{1}{r}\left\{\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right)+\frac{\partial}{\partial\varphi}\left(\frac{1}{r}\frac{\partial\psi}{\partial\varphi}\right)+\frac{\partial}{\partial z}\left(r\frac{\partial\psi}{\partial z}\right)\right\} \\
&= \frac{\partial^2\psi}{\partial r^2}+\frac{1}{r}\frac{\partial\psi}{\partial r}+\frac{1}{r}\frac{\partial\psi}{\partial\varphi}+\frac{\partial^2\psi}{\partial z^2}.
\end{aligned}
$$

(3)

球座標におけるラプラシアンは

$$
\begin{aligned}
\nabla^2\psi &= \frac{1}{r^2\sin\theta}\left\{\frac{\partial}{\partial r}\left(r^2\sin\theta\frac{\partial\psi}{\partial r}\right)+\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{\partial}{\partial\varphi}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\varphi}\right)\right\} \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right)+\frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right)+\frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\varphi^2}
\end{aligned}
$$