演習問題 12.6 解答例

(2)

まず次のような $f(z)$ について考える

$$
\begin{aligned}
f(z) &= \frac{e^{ibz}}{z^4+a^4} \\
&= \frac{e^{ibz}}{(z-ae^{\frac{\pi}{4}i})(z+ae^{\frac{\pi}{4}i})(z-ae^{\frac{3}{4}\pi i})(z+ae^{\frac{3}{4}\pi i})}.
\end{aligned}
$$

$\mathrm{Im}[z]>0$ において $f(z)$ は $z=ae^{\frac{\pi}{4}i}, ae^{\frac{3}{4}\pi i}$ で特異点を持つ. このときコーシーの積分定理, 及び留数定理から

$$
\begin{aligned}
\int_{-\infty}^{\infty}\frac{e^{ibz}}{z^4+a^4}dz &= 2\pi i\{\mathrm{Res}f(ae^{\frac{\pi}{4}i})+\mathrm{Res}f(ae^{\frac{3}{4}\pi i})\} \\
&= 2\pi i\left\{\frac{e^{-\frac{ab}{\sqrt{2}}}e^{\frac{ab}{\sqrt{2}}i}}{2\sqrt{2}a^3(-1+i)}+\frac{e^{-\frac{ab}{\sqrt{2}}}e^{-\frac{ab}{\sqrt{2}}i}}{2\sqrt{2}a^3(1+i)}\right\} \\
&= \frac{\pi}{\sqrt{2}a^3}e^{-\frac{ab}{\sqrt{2}}}\left(\frac{e^{\frac{ab}{\sqrt{2}}i}}{1+i}+\frac{e^{-\frac{ab}{\sqrt{2}}i}}{1-i}\right) \\
&= \frac{\pi}{\sqrt{2}a^3}e^{-\frac{ab}{\sqrt{2}}}\left\{\frac{1}{2}\left(e^{\frac{ab}{\sqrt{2}}i}+e^{-\frac{ab}{\sqrt{2}}i}\right)+\frac{i}{2}\left(e^{-\frac{ab}{\sqrt{2}}i}-e^{\frac{ab}{\sqrt{2}}i}\right)\right\} \\
&= \frac{\pi}{\sqrt{2}a^3}e^{-\frac{ab}{\sqrt{2}}}\left(\cos\frac{ab}{\sqrt{2}}+\sin\frac{ab}{\sqrt{2}}\right).
\end{aligned}
$$

ここで

$$
\int_{-\infty}^{\infty}\frac{e^{ibz}}{z^4+a^4}dz = \int_{-\infty}^{\infty}\frac{\cos bz}{z^4+a^4}dz + i\int_{-\infty}^{\infty}\frac{\sin bz}{z^4+a^4}dz
$$

であり, 実部と虚部は独立なので,

$$
\int_{-\infty}^{\infty}\frac{\cos bx}{x^4+a^4}dx = \frac{\pi}{\sqrt{2}a^3}e^{-\frac{ab}{\sqrt{2}}}\left(\cos\frac{ab}{\sqrt{2}}+\sin\frac{ab}{\sqrt{2}}\right).
$$

同様に次のような $g(z)$ について考える

$$
\begin{aligned}
g(z) &= \frac{ze^{ibz}}{z^4+a^4} \\
&= \frac{ze^{ibz}}{(z-ae^{\frac{\pi}{4}i})(z+ae^{\frac{\pi}{4}i})(z-ae^{\frac{3}{4}\pi i})(z+ae^{\frac{3}{4}\pi i})}.
\end{aligned}
$$

$\mathrm{Im}[z] > 0$ において $g(z)$ は $z = ae^{\frac{\pi}{4}i}, ae^{\frac{3}{4}\pi i}$ で特異点を持つ. このときコーシーの積分定理, 及び留数定理から

$$
\begin{aligned}
\int_{-\infty}^{\infty} \frac{ze^{ibz}}{z^4+a^4} dz &= 2\pi i \{ \mathrm{Res} g(ae^{\frac{\pi}{4}i}) + \mathrm{Res} g(ae^{\frac{3}{4}\pi i}) \} \\
&= 2\pi i \left\{ \frac{(1+i)e^{-\frac{ab}{\sqrt{2}}} e^{\frac{ab}{\sqrt{2}}i}}{4a^2(-1+i)} + \frac{(-1+i)e^{-\frac{ab}{\sqrt{2}}} e^{-\frac{ab}{\sqrt{2}}i}}{4a^2(1+i)} \right\} \\
&= \frac{\pi}{2a^2} e^{-\frac{ab}{\sqrt{2}}} \left(e^{\frac{ab}{\sqrt{2}}i} - e^{-\frac{ab}{\sqrt{2}}i} \right) \\
&= \frac{\pi}{a^2} e^{-\frac{ab}{\sqrt{2}}} \left\{ \frac{1}{2} \left(e^{\frac{ab}{\sqrt{2}}i} - e^{-\frac{ab}{\sqrt{2}}i} \right) \right\} \\
&= i \frac{\pi}{a^2} e^{-\frac{ab}{\sqrt{2}}} \sin \frac{ab}{\sqrt{2}}.
\end{aligned}
$$

ここで

$$
\int_{-\infty}^{\infty} \frac{ze^{ibz}}{z^4+a^4} dz = \int_{-\infty}^{\infty} \frac{z \cos bz}{z^4+a^4} dz + i \int_{-\infty}^{\infty} \frac{z \sin bz}{z^4+a^4} dz
$$

であり, 実部と虚部は独立なので,

$$
\int_{-\infty}^{\infty} \frac{x \cos bx}{x^4+a^4} dx = \frac{\pi}{a^2} e^{-\frac{ab}{\sqrt{2}}} \sin \frac{ab}{\sqrt{2}}.
$$