Hamiltonian Fluid Mechanics

(Classical Mechanics applied to Fluids)

Important caveat:
Hamiltonian methods add *nothing* to the actual physics.

So why do it?

It is beautiful—simple, geometric.
It makes things *easier to do* and *easier to understand*.

Examples of things that are easier to do:

1. Find conservation laws.
2. Construct approximations that preserve conservation laws.
3. Choose variables for which the physics takes the simplest mathematical form.

Examples of things that are easier to understand:

1. Why potential vorticity is peculiar to fluids.
2. Why wave action is conserved.
3. How potential energy is related to available potential energy.

Important theme: Conservation Laws ⇔ Symmetry Properties

This will be an *elementary introduction* following my book:

Lectures on Geophysical Fluid Dynamics, Oxford, 1998
Chapter 1 (pp 1-12), Chapter 4 (pp 197-206), **Chapter 7**

Quick review of basic classical mechanics

Point masses moving in three-dimensional space:

$$\mathbf{v}_i = \frac{d}{dt}\mathbf{x}_i(t)$$

Kinetic energy $\quad T = \tfrac{1}{2}\sum_i m_i \frac{d\mathbf{x}_i}{dt}\cdot\frac{d\mathbf{x}_i}{dt}$

Potential energy $\quad V = V(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$

Lagrangian $\quad L = T - V$

Action $\quad A\left[\mathbf{x}_1(t), \mathbf{x}_2(t), \ldots, \mathbf{x}_N(t)\right] = \int_{t_1}^{t_2} L \; dt$

Hamilton's principle:

$$\delta\int_{t_1}^{t_2} L \; dt = 0$$ for *arbitrary* $\delta\mathbf{x}_i(t)$ with $\delta\mathbf{x}_i(t_1) = \delta\mathbf{x}_i(t_2) = 0$

For our example:

$$L = \tfrac{1}{2}\sum_i m_i \frac{d\mathbf{x}_i}{dt}\cdot\frac{d\mathbf{x}_i}{dt} - V(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$$

$$\delta\int L\,dt = \int dt\left\{\sum_i \tfrac{1}{2}m_i \frac{d(\mathbf{x}_i + \delta\mathbf{x}_i)}{dt}\cdot\frac{d(\mathbf{x}_i + \delta\mathbf{x}_i)}{dt} - V(\mathbf{x}_1 + \delta\mathbf{x}_1, \mathbf{x}_2 + \delta\mathbf{x}_2, \ldots, \mathbf{x}_N + \delta\mathbf{x}_N)\right\}$$
$$- \tfrac{1}{2}\sum_i m_i \frac{d\mathbf{x}_i}{dt}\cdot\frac{d\mathbf{x}_i}{dt} - V(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_N)$$

$$= \int dt \sum_i \left\{ m_i \frac{d\mathbf{x}_i}{dt}\cdot\frac{d\delta\mathbf{x}_i}{dt} - \frac{\partial V}{\partial\mathbf{x}_i}\cdot\delta\mathbf{x}_i\right\} + O(\delta\mathbf{x}^2)$$

$$= \int dt \sum_i \left\{ -m_i \frac{d^2\mathbf{x}_i}{dt^2} - \frac{\partial V}{\partial\mathbf{x}_i}\right\}\cdot\delta\mathbf{x}_i$$

$$\Rightarrow \quad m_i \frac{d^2\mathbf{x}_i}{dt^2} = -\frac{\partial V}{\partial\mathbf{x}_i}$$

A fluid is a *continuous distribution of particles.*

$$\mathbf{x}_i(t) \Rightarrow \mathbf{x}(a,b,c,\tau)$$

where

$$(a,b,c) = \mathbf{a}$$

are *particle labels* assigned such that

$$d\mathbf{a} = d(\text{mass})$$

and

$$t = \tau \qquad \text{so that} \qquad \frac{\partial F}{\partial \tau} \equiv \frac{DF}{Dt}$$

Note: $\dfrac{\partial \mathbf{a}}{\partial \tau} = 0$ (particle labels are conserved).

For any $F = F(x,y,z,t) = F(a,b,c,\tau)$ the chain rule implies:

$$\frac{\partial F}{\partial \tau} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial \tau} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial \tau} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial \tau} + \frac{\partial F}{\partial t}\frac{\partial t}{\partial \tau}$$

$$\Leftrightarrow \frac{DF}{Dt} = u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} + w\frac{\partial F}{\partial z} + \frac{\partial F}{\partial t}$$

Continuity Equation

Because $d\mathbf{a} = d(\text{mass})$, the *mass density*

$$\rho = \frac{d(\text{mass})}{d(\text{volume})} = \frac{\partial(a,b,c)}{\partial(x,y,z)}$$

and *specific volume*

$$\alpha = \frac{1}{\rho} = \frac{\partial(x,y,z)}{\partial(a,b,c)}$$

This implies

$$\begin{aligned}
\frac{\partial\alpha}{\partial\tau} &= \frac{\partial(u,y,z)}{\partial(a,b,c)} + \frac{\partial(x,v,z)}{\partial(a,b,c)} + \frac{\partial(x,y,w)}{\partial(a,b,c)} \\
&= \frac{\partial(x,y,z)}{\partial(a,b,c)}\left[\frac{\partial(u,y,z)}{\partial(x,y,z)} + \frac{\partial(x,v,z)}{\partial(x,y,z)} + \frac{\partial(x,y,w)}{\partial(x,y,z)}\right] \\
&= \alpha\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right]
\end{aligned}$$

which is equivalent to

$$\frac{\partial\rho}{\partial\tau} = -\rho\left[\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right]$$

The continuity equation is a built-in feature of the Lagrangian description.

Hamilton's principle for the fluid

Using the analogy with particle mechanics:

$$\tfrac{1}{2}\sum_i m_i \frac{d\mathbf{x}_i}{dt}\cdot\frac{d\mathbf{x}_i}{dt} \rightarrow \tfrac{1}{2}\iiint d\mathbf{a}\,\frac{\partial\mathbf{x}}{\partial\tau}\cdot\frac{\partial\mathbf{x}}{\partial\tau}$$

$$V(\mathbf{x}_1,\mathbf{x}_2,\ldots,\mathbf{x}_N) \rightarrow \iiint d\mathbf{a}\left\{E(\alpha,S)+\Phi(\mathbf{x})\right\}$$

$E(\alpha,S)$ = internal energy per unit mass

$$\alpha = \frac{\partial(x,y,z)}{\partial(a,b,c)} = \text{volume per unit mass}$$

$S(a,b,c)$ = entropy per unit mass. ($\partial S/\partial\tau = 0$)

$$L[\mathbf{x}(\mathbf{a},\tau)] = \iiint d\mathbf{a}\left\{\tfrac{1}{2}\frac{\partial\mathbf{x}}{\partial\tau}\cdot\frac{\partial\mathbf{x}}{\partial\tau} - E\left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})},S(\mathbf{a})\right)-\Phi(\mathbf{x})\right\}$$

$E(\alpha,S)$ is a prescribed function of its 2 arguments.
This is called the *fundamental relation* of thermodynamics (Gibbs, Tisza, Callendar). The fundamental relation determines all other thermodynamic variables.

Hamilton's principle states:

$$\delta \int d\tau \iiint d\mathbf{a} \left\{ \tfrac{1}{2} \frac{\partial \mathbf{x}}{\partial \tau} \cdot \frac{\partial \mathbf{x}}{\partial \tau} - E\left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}, S(\mathbf{a}) \right) - \Phi(\mathbf{x}) \right\} = 0$$

for arbitrary $\delta \mathbf{x}(\mathbf{a}, \tau)$

The **essence** of a fluid is that the **a-**derivatives of **x** enter *L* only through the Jacobian

$$\frac{\partial(x, y, z)}{\partial(a, b, c)}$$

This fact is responsible for all the *distinctive* properties of fluid mechanics:

1. The existence of an Eulerian description

2. The importance of vorticity

3. The conservation of potential vorticity

Working out Hamilton's principle:

$$\delta \int d\tau \iiint d\mathbf{a} \left\{ \tfrac{1}{2} \frac{\partial \mathbf{x}}{\partial \tau} \cdot \frac{\partial \mathbf{x}}{\partial \tau} - E\left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}, S(\mathbf{a}) \right) - \Phi(\mathbf{x}) \right\}$$

$$= \int d\tau \iiint d\mathbf{a} \left\{ -\frac{\partial^2 \mathbf{x}}{\partial \tau^2} \cdot \delta\mathbf{x} - \frac{\partial E(\alpha, S)}{\partial \alpha} \delta \frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} - \frac{\partial \Phi}{\partial \mathbf{x}} \cdot \delta\mathbf{x} \right\}$$

With the help of a useful identity:

$$\iiint d\mathbf{a} \quad F \quad \delta \frac{\partial(x,y,z)}{\partial(a,b,c)}$$

$$= \iiint d\mathbf{a} \quad F \quad \left\{ \frac{\partial(\delta x, y, z)}{\partial(a,b,c)} + \frac{\partial(x, \delta y, z)}{\partial(a,b,c)} + \frac{\partial(x, y, \delta z)}{\partial(a,b,c)} \right\}$$

$$= \iiint d\mathbf{a} \quad F \quad \frac{\partial(x,y,z)}{\partial(a,b,c)} \left\{ \frac{\partial(\delta x, y, z)}{\partial(x,y,z)} + \frac{\partial(x, \delta y, z)}{\partial(x,y,z)} + \frac{\partial(x, y, \delta z)}{\partial(x,y,z)} \right\}$$

$$= \iiint d\mathbf{x} \quad F \quad \{ \nabla \cdot \delta\mathbf{x} \}$$

$$= -\iiint d\mathbf{x} \quad \nabla F \cdot \delta\mathbf{x} + \oiint \quad F \quad \delta\mathbf{x} \cdot \mathbf{n}$$

Gives

$$\int d\tau \iiint d\mathbf{x} \quad \rho \quad \left\{ -\frac{\partial^2 \mathbf{x}}{\partial \tau^2} - \frac{1}{\rho} \frac{\partial p}{\partial \mathbf{x}} - \frac{\partial \Phi}{\partial \mathbf{x}} \right\} \cdot \delta\mathbf{x} + \oiint d\sigma \quad p \quad \delta\mathbf{x} \cdot \mathbf{n}$$

where $\qquad p \equiv -\dfrac{\partial E(\alpha, S)}{\partial \alpha}$

The surface integral vanishes if the surface is rigid ($\delta\mathbf{x} \cdot \mathbf{n}$) or free ($p$=0)

Thus Hamilton's principle says that

$$\frac{\partial^2 \mathbf{x}}{\partial \tau^2} + \frac{1}{\rho}\frac{\partial p}{\partial \mathbf{x}} + \frac{\partial \Phi}{\partial \mathbf{x}} = 0 \qquad \Rightarrow \qquad \frac{D\mathbf{v}}{Dt} + \frac{1}{\rho}\nabla p + \nabla \Phi = 0$$

To this we may add:

$$\rho \equiv \frac{\partial(a,b,c)}{\partial(x,y,z)} \qquad \Rightarrow \qquad \frac{D\rho}{Dt} + \rho \nabla \cdot \mathbf{v} = 0$$

$$\frac{\partial}{\partial \tau} S(a,b,c) = 0 \qquad \Rightarrow \qquad \frac{DS}{Dt} = 0$$

$$p \equiv -\frac{\partial E(\alpha,S)}{\partial \alpha} \qquad \Rightarrow \qquad p = p(\rho,S)$$

These are the general equations of a perfect fluid.

Once again, the **essence of a fluid** is that the derivatives

$$\frac{\partial x_i}{\partial a_j}$$

enter the Lagrangian only through the Jacobian

$$\frac{\partial(x,y,z)}{\partial(a,b,c)}$$

[Note: $(x_1,x_2,x_3)\equiv(x,y,z)$ and $(a_1,a_2,a_3)\equiv(a,b,c)$.]

This leads to a symmetry property that corresponds to the relabeling of fluid particles in such a way that the Jacobian is not affected.

First suppose <u>S=uniform</u>, and consider a *particle relabeling*

$$\begin{aligned} a' &= a+\delta a(a,b,c,\tau) \\ b' &= b+\delta b(a,b,c,\tau) \\ c' &= c+\delta c(a,b,c,\tau) \end{aligned}$$

for which

$$\frac{\partial(a',b',c')}{\partial(x,y,z)}=\frac{\partial(a,b,c)}{\partial(x,y,z)}$$

This implies

$$\frac{\partial\,\delta a}{\partial a}+\frac{\partial\,\delta b}{\partial b}+\frac{\partial\,\delta c}{\partial c}=0$$

i.e.

$$\delta\mathbf{a}=\nabla_{\mathbf{a}}\times\delta\mathbf{T}(\mathbf{a},\tau)$$

Since

$$\text{Action} = \int d\tau \iiint d\mathbf{a} \left\{ \tfrac{1}{2} \frac{\partial \mathbf{x}}{\partial \tau} \cdot \frac{\partial \mathbf{x}}{\partial \tau} - E\left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} \right) - \Phi(\mathbf{x}) \right\}$$

such a variation implies

$$\delta(\text{Action}) = \int d\tau \iiint d\mathbf{a} \left\{ \frac{\partial \mathbf{x}}{\partial \tau} \cdot \delta \frac{\partial \mathbf{x}}{\partial \tau} \right\}$$

Here

$$\delta \frac{\partial \mathbf{x}}{\partial \tau} = \left. \frac{\partial \mathbf{x}}{\partial \tau} \right|_{\mathbf{a}'} - \left. \frac{\partial \mathbf{x}}{\partial \tau} \right|_{\mathbf{a}}$$

is the change in the time derivative caused by holding fixed a *different* set of labels.

We compute:

$$\begin{aligned} \left. \frac{\partial \mathbf{x}}{\partial \tau} \right|_{\mathbf{a}'} &= \left. \frac{\partial \mathbf{x}}{\partial \tau} \right|_{\mathbf{a}} + \left. \frac{\partial \mathbf{x}}{\partial a} \frac{\partial a}{\partial \tau} \right|_{\mathbf{a}'} + \left. \frac{\partial \mathbf{x}}{\partial b} \frac{\partial b}{\partial \tau} \right|_{\mathbf{a}'} + \left. \frac{\partial \mathbf{x}}{\partial c} \frac{\partial c}{\partial \tau} \right|_{\mathbf{a}'} \\ &= \left. \frac{\partial \mathbf{x}}{\partial \tau} \right|_{\mathbf{a}} - \left. \frac{\partial \mathbf{x}}{\partial a} \frac{\partial \delta a}{\partial \tau} \right|_{\mathbf{a}'} - \left. \frac{\partial \mathbf{x}}{\partial b} \frac{\partial \delta b}{\partial \tau} \right|_{\mathbf{a}'} - \left. \frac{\partial \mathbf{x}}{\partial c} \frac{\partial \delta c}{\partial \tau} \right|_{\mathbf{a}'} \end{aligned}$$

Thus

$$\delta \frac{\partial \mathbf{x}}{\partial \tau} = - \left. \frac{\partial \mathbf{x}}{\partial a} \frac{\partial \delta a}{\partial \tau} \right|_{\mathbf{a}} - \left. \frac{\partial \mathbf{x}}{\partial b} \frac{\partial \delta b}{\partial \tau} \right|_{\mathbf{a}} - \left. \frac{\partial \mathbf{x}}{\partial c} \frac{\partial \delta c}{\partial \tau} \right|_{\mathbf{a}} + O\left(\delta \mathbf{a}^2\right)$$

and hence

$$\delta(\text{Action}) = -\int d\tau \iiint d\mathbf{a} \; \frac{\partial x_i}{\partial \tau} \frac{\partial x_i}{\partial a_j} \frac{\partial}{\partial \tau} \delta a_j = \cdots$$

$$\delta(\text{Action}) = \int d\tau \iiint d\mathbf{a} \; \frac{\partial}{\partial\tau}\left(\frac{\partial x_i}{\partial\tau}\frac{\partial x_i}{\partial a_j}\right) \; \delta a_j$$

Define $A_j \equiv \frac{\partial x_i}{\partial\tau}\frac{\partial x_i}{\partial a_j}$ and use $\delta\mathbf{a} = \nabla_{\mathbf{a}} \times \delta\mathbf{T}(\mathbf{a},\tau)$

Then

$$\delta A = \int d\tau \iiint d\mathbf{a} \; \frac{\partial}{\partial\tau}(\nabla_{\mathbf{a}} \times \mathbf{A}) \cdot \delta\mathbf{T} = 0$$

In summary: For the homentropic case, the particle-relabeling symmetry property leads to the fundamental conservation law

$$\frac{\partial}{\partial\tau}(\nabla_{\mathbf{a}} \times \mathbf{A}) = 0$$

where

$$\nabla_{\mathbf{a}} \equiv \left(\frac{\partial}{\partial a},\frac{\partial}{\partial b},\frac{\partial}{\partial c}\right) \equiv \left(\frac{\partial}{\partial a_1},\frac{\partial}{\partial a_2},\frac{\partial}{\partial a_3}\right)$$

is the gradient operator in label-space, and

$$\mathbf{A} \equiv (A_1, A_2, A_3) = u\nabla_{\mathbf{a}}x + v\nabla_{\mathbf{a}}y + w\nabla_{\mathbf{a}}z$$

is the "velocity measured in Lagrangian coordinates."

This is law the most general statement of vorticity conservation for homentropic flow. It implies Ertel's theorem, Kelvin's theorem, helicity conservation, and any other vorticity theorem you can think of!

Rule #1: <u>Every result obtained by using Hamiltonian methods can also be obtained without using Hamiltonian methods.</u>

This means we must be able to derive the general vorticity theorem

$$\frac{\partial}{\partial \boldsymbol{\tau}}(\nabla_{\mathbf{a}} \times \mathbf{A}) = 0$$

directly from the equations of motion. It is instructive to do this!

The curl of the momentum equation gives the well-known vorticity equation

$$\frac{D}{Dt}\omega + \omega(\nabla \cdot \mathbf{v}) = (\omega \cdot \nabla)\mathbf{v} + \nabla p \times \nabla\left(\frac{1}{\rho}\right) \qquad \omega \equiv \nabla \times \mathbf{v}$$

Combining this with the continuity equation gives

$$\frac{D}{Dt}\left(\frac{\omega}{\rho}\right) = \left(\frac{\omega}{\rho} \cdot \nabla\right)\mathbf{v} + \frac{1}{\rho}\nabla p \times \nabla\left(\frac{1}{\rho}\right)$$

In the homentropic case, $p = p(\rho)$, so

$$\frac{D}{Dt}\left(\frac{\omega}{\rho}\right) = \left(\frac{\omega}{\rho} \cdot \nabla\right)\mathbf{v}$$

This is the *same* equation as for an infinitesimal displacement vector $\delta\mathbf{r}$ between fluid particles:

$$\frac{d}{dt}\delta\mathbf{r} = (\delta\mathbf{r} \cdot \nabla)\mathbf{v}$$

If θ is any conserved scalar,

$$\frac{D\theta}{Dt} = 0$$

then

$$\frac{d}{dt}(\delta\mathbf{r} \cdot \nabla\theta) = 0$$

By the analogy between $\frac{\omega}{\rho}$ and $\delta\mathbf{r}$ it follows that

$$\frac{D}{Dt}\left(\frac{\omega}{\rho} \cdot \nabla\theta\right) = 0$$

This is Ertel's theorem for a homentropic fluid.

Now let

$$\theta_1(\mathbf{x},t),\ \theta_2(\mathbf{x},t),\ \theta_3(\mathbf{x},t)$$

be three *independent* scalars satisfying:

$$\frac{D\theta_1}{Dt} = 0, \qquad \frac{D\theta_2}{Dt} = 0, \qquad \frac{D\theta_3}{Dt} = 0$$

Then $$\frac{D}{Dt}(Q_1,Q_2,Q_3) = 0 \qquad \text{where} \qquad Q_i = \frac{\omega}{\rho} \cdot \nabla\theta_i$$

Think of

$$\nabla\theta_1,\ \nabla\theta_2,\ \nabla\theta_3$$

as basis vectors.

If

$$\mathbf{v} = A_1 \nabla \theta_1 + A_2 \nabla \theta_2 + A_3 \nabla \theta_3$$

and

$$d\theta_1\, d\theta_2\, d\theta_3 = d(\text{mass})$$

Then

$$\mathbf{Q} = \nabla_\theta \times \mathbf{A}$$

Identifying $\theta_1, \theta_2, \theta_3$ with a_1, a_2, a_3 we have

$$\frac{\partial}{\partial \boldsymbol{\tau}} \left(\nabla_{\mathbf{a}} \times \mathbf{A} \right) = 0$$

This is the *same* conservation law that was derived *more directly* from Hamilton's principle and the particle-relabeling symmetry! There:

$$\mathbf{A} \equiv \left(A_1, A_2, A_3 \right) = u \nabla_{\mathbf{a}} x + v \nabla_{\mathbf{a}} y + w \nabla_{\mathbf{a}} z$$

For homentropic fluid, the conserved **Q** is just "ordinary vorticity measured in Lagrangian coordinates."

For the *non*-homentropic case

$$\frac{D}{Dt}\left(\frac{\omega}{\rho} \cdot \nabla \theta \right) = \frac{1}{\rho^3} \nabla \theta \cdot \left(\nabla \rho \times \nabla p \right)$$

Since

$$p = p(\rho, S)$$

we have

$$\frac{D}{Dt}\left(\frac{\omega}{\rho} \cdot \nabla S \right) = 0$$

To obtain the non-homentropic result from symmetry:

Note that since

$$E = E\left(\frac{\partial(\mathbf{x})}{\partial(\mathbf{a})}, S(\mathbf{a})\right)$$

the particle-label variations must satisfy

$$\delta \ \frac{\partial(\mathbf{x})}{\partial(\mathbf{a})} = 0$$

and

$$\delta \ S(\mathbf{a}) = 0$$

The easiest way to do this is to choose $S=c$. Then we find that

$$\delta a = -\frac{\partial}{\partial b}\delta\psi(\mathbf{a},\tau), \quad \delta b = \frac{\partial}{\partial a}\delta\psi(\mathbf{a},\tau), \quad \delta c = 0$$

instead of

$$\boldsymbol{\delta}\mathbf{a} = \nabla_{\mathbf{a}} \times \boldsymbol{\delta}\mathbf{T}(\mathbf{a},\tau)$$

The more restricted variation leads to

$$\frac{\partial}{\partial\tau}\left[(\nabla_{\mathbf{a}} \times \mathbf{A}) \cdot \nabla_{\mathbf{a}} c\right] = \frac{\partial}{\partial\tau}\left[\frac{(\nabla \times \mathbf{v}) \cdot \nabla S}{\rho}\right] = 0$$

Overall summary:

In homentropic fluid, the particle relabeling symmetry leads to the conservation law

$$\frac{\partial}{\partial \boldsymbol{\tau}}\left(\nabla_{\mathbf{a}} \times \mathbf{A}\right) = 0$$

where

$$\mathbf{A} \equiv (A,B,C) = u\nabla_{\mathbf{a}} x + v\nabla_{\mathbf{a}} y + w\nabla_{\mathbf{a}} z$$

or, equivalently,

$$\mathbf{v} \equiv (u,v,w) = A\nabla_{\mathbf{x}} a + B\nabla_{\mathbf{x}} b + C\nabla_{\mathbf{x}} c$$

The conserved potential vorticity is a vector.

In the nonhomentropic case,

$$\nabla_{\mathbf{a}} S(a,b,c) \cdot \frac{\partial}{\partial \boldsymbol{\tau}}\left(\nabla_{\mathbf{a}} \times \mathbf{A}\right) = \frac{\partial}{\partial \boldsymbol{\tau}}\left[\left(\nabla_{\mathbf{a}} \times \mathbf{A}\right) \cdot \nabla_{\mathbf{a}} S\right] = 0$$

The conserved potential vorticity is a scalar.

Corollaries

The (homentropic) conservation law

$$\frac{\partial}{\partial \boldsymbol{\tau}}\left(\nabla_{\mathbf{a}} \times \mathbf{A}\right) = 0$$

has many consequences.

These include:

Kelvin's theorem:

$$\frac{d}{dt}\oint \mathbf{A} \cdot d\mathbf{a} = \frac{d}{dt}\oint \mathbf{v} \cdot d\mathbf{x} = 0$$

Helicity conservation:

$$\frac{d}{dt}\iiint d\mathbf{a}\;\; \mathbf{A} \cdot \left(\nabla_{\mathbf{a}} \times \mathbf{A}\right) = \frac{d}{dt}\iiint d\mathbf{x}\;\; \mathbf{v} \cdot \left(\nabla_{\mathbf{x}} \times \mathbf{v}\right) = 0$$