Fundamentals of magnetohydrodynamics

Consider an electrically conducting fluid with conductivity σ, moving with a (local) velocity v << c, for which $\varepsilon=\varepsilon_o$, $\mu=\mu_o$.

$$\nabla \times \mathbf{B} = \mu_o \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \qquad \nabla \cdot \mathbf{B} = 0$$

Maxwell‘s law (Ampère‘s law)

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \nabla \cdot \mathbf{E} = \frac{\xi}{\varepsilon_o}$$

Faraday‘s law

$$\mathbf{B}' = \mathbf{B} - \frac{\mathbf{v}\times\mathbf{E}}{c^2} \qquad \mathbf{E}' = \mathbf{E} + \mathbf{v}\times\mathbf{B}$$

Transformation rules into moving reference frame (valid for v << c)

$$\mathbf{j} = \sigma \mathbf{E}$$

Ohm‘s law for fluid at rest

Equations in the MHD approximation: Estimate relative magnitude of various terms. The ratio of typical length scale L to typical time scale T of the system V = L/T << c. From Faraday‘s law we obtain E/L ~ B/T $\Rightarrow$ E ~ VB. Using this, the last term in Maxwell‘s law is of order $VB/(c^2T)$ and the first term of order B/L $\Rightarrow$ their ratio is $V^2/c^2 << 1$ and the last term (displacement current) can be neglected. Similarly it can be shown that B‘ ≈ B. The electrical current in a **moving** conductor is obtained as **j** = σ**E‘** + **v**ξ. The scaling analysis shows again that the last term relative to the first is of order vV/c^2, hence negligible.

Symbols: σ – elctrical conductivity, μ – magnetic permeability, ε – dielectric constant, $c=(\mu_o\varepsilon_o)^{-1/2}$ – speed of light, B – magnetic induction („magnetic field“), E – electric field, j – current density, ξ – electric charge density

Induction equation

$$\nabla \times \mathbf{B} = \mu_o \mathbf{j} \qquad \nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \qquad \mathbf{j} = \sigma(\mathbf{E} + \mathbf{v} \times \mathbf{B})$$

MHD-equations (pre-Maxwell equations and generalized Ohm's law.

Eliminate j, introduce magnetic diffusivity $\lambda=(\mu\sigma)^{-1}$, take curl: $\nabla\times(\lambda\nabla\times\mathbf{B}) = \nabla\times\mathbf{E} + \nabla\times(\mathbf{v}\times\mathbf{B})$.
Eliminate E. For $\nabla\cdot\mathbf{B}=0 \Rightarrow \nabla\times\nabla\times\mathbf{B} = -\nabla^2\mathbf{B}$, so for λ=const:

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \lambda \nabla^2 \mathbf{B}$$

Induction equation

$$\frac{\partial \mathbf{B}}{\partial t} + \underbrace{(\mathbf{v}\cdot\nabla)\mathbf{B}}_{\text{Advection}} = \underbrace{(\mathbf{B}\cdot\nabla)\mathbf{v}}_{\text{Induction}} + \underbrace{\lambda\nabla^2\mathbf{B}}_{\text{Diffusion}}$$

For $\nabla\cdot\mathbf{v} = 0$

For a given velocity field (magnetokinematics) and initial condition and boundary conditions for B, the evolution for B can be calculated. The search of velocity fields that lead to solutions with non-decaying B is subject of the **kinematic dynamo theory**.

Symbols: λ – magnetic diffusivity

Induction equation

Scale magnetic field and velocity by characteristic values B_o, U, length by D, and time by D/U. The equation in terms of dimensionless variables is

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \frac{1}{Rm} \nabla^2 \mathbf{B} \qquad Rm = \frac{UD}{\lambda}$$

Magnetic Reynolds number

1) For small Rm diffusion dominates, for large Rm advection and induction dominate. Working dynamos require Rm>10 at least.
2) Because $\nabla \cdot \mathbf{B}=0$, the magnetic field can be decomposed in toroidal and poloidal parts. In a sphere, the toroidal field has no radial component and the poloidal field has no associated radial current ($j_r \sim [\nabla \times \mathbf{B}]_r$). The toroidal field cannot be observed outside the sphere.
3) If the region outside the sphere is insulating, $\nabla \times \mathbf{B} = 0$, $\nabla \cdot \mathbf{B} = 0$, $\nabla^2 \mathbf{B} = 0$. The exterior field is poloidal and can be represented as gradient $\mathbf{B} = -\nabla \Phi$. Appropriate boundary condition at the surface of the sphere is a matching condition of the internal field to a potential external field that decays with radius if there are no external sources to the field. Φ is expanded in spherical harmonic functions.
   (n=1 dipole, n=2 quadrupole)

$$\Phi = a \sum_{n=1}^{\infty} \left(\frac{a}{r}\right)^{n+1} \sum_{m=0}^{n} (g_n^m \cos m\varphi + h_n^m \sin m\varphi)\; P_n^m(\cos\theta)$$

   An equivalent decomposition in spherical harmonic functions can be made for the toroidal and poloidal vector potentials of the field inside the sphere, but with arbitrary radial functions $\Pi_{nm}(r)$, $\Theta_{nm}(r)$.

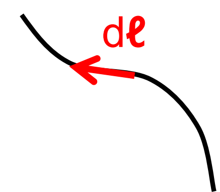
Symbols: Rm – magnetic Reynolds number, Φ –scalar magnetic potential, a – reference (Earth) radius, g_n^m, h_n^m – Gauss coefficients, P_n^m – associate Legendre functions, n – harmonic degree, m - harmonic order, Π,Θ – vector magnetic potent

Decay modes / Alfvén's theorem

At low Rm << 1 we can neglect the induction/advection term: $\partial \mathbf{B}/\partial t = \lambda \nabla^2 \mathbf{B}$. Without external sources, any field will decay. Scaling analysis tells that the basic time scale for this is D^2/λ. In a sphere exponential decay is found for particular field configurations called **free decay modes**. The slowest decay occurs for a certain dipolar poloidal field, with decay time of $\tau = R^2/(\pi^2 \lambda)$. Earth's core $\lambda \approx 1\ m^2/s$, R=3,480 km $\Rightarrow \tau \approx$ 40,000 yr.

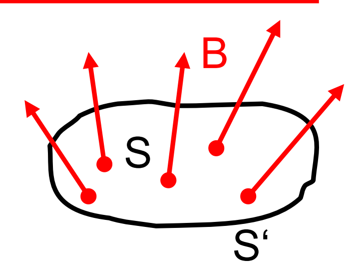
At very large Rm >> 1, we neglect the diffusion term: $\partial \mathbf{B}/\partial t + (\mathbf{v} \cdot \nabla)\mathbf{B} = (\mathbf{B} \cdot \nabla)\mathbf{v}$. The term on the left side is the change of B in a Lagrangian reference frame, $D\mathbf{B}/Dt = (\mathbf{B} \cdot \nabla)\mathbf{v}$.

Consider a line connecting particles of the moving fluid. $d\boldsymbol{\ell}$ is a vector along this line connecting two close particles). When **v** is the velocity at the starting point, that at the end-point is $\mathbf{v} + (d\boldsymbol{\ell} \cdot \nabla)\mathbf{v}$. Therefore the change of the (moving) vector is $D\boldsymbol{\ell}/Dt = (\boldsymbol{\ell} \cdot \nabla)\mathbf{v}$. This equation is formally identical to that governing **B**. A magnetic field line will pass through the same fluid particles at all times.



Alfvén's theorem: When diffusion is negligible, the magnetic field is „frozen" into the fluid.

Consider a closed material line S' that moves with the flow. Magnetic field lines that pass through the surface S bounded by S' will pass through this surface at all times. In particular, the magnetic flux $\int \mathbf{B} \cdot d\mathbf{S}$ through this (moving and deforming) surface will remain the same at all times.

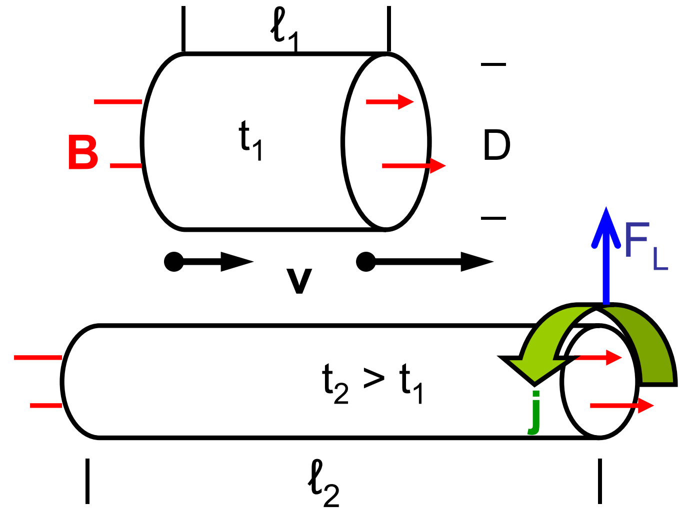


Magnetic flux is „frozen" into a material surface.

Symbols: $D/Dt = \partial/\partial t + v \cdot \nabla$ - Lagrangian time derivative, $d\boldsymbol{\ell}$ – line element

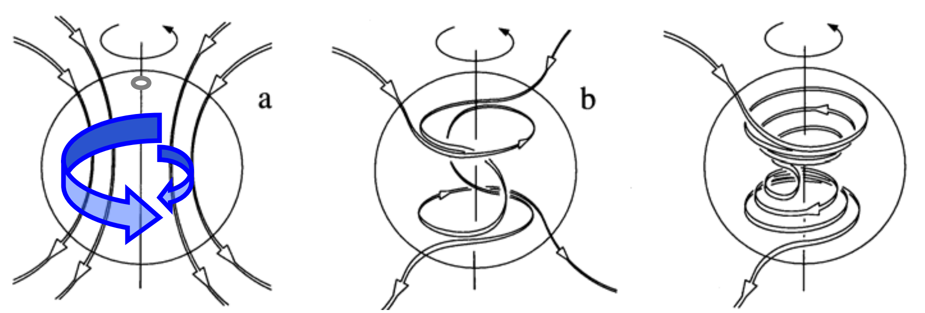
Field line stretching

For negligible diffusion $D\mathbf{B}/Dt = (\mathbf{B}\cdot\nabla)\mathbf{v}$. If a material line (magnetic field line) is elongated by the flow, the magnetic field strength will increase. Consider a cylinder penetrated by a bundle of magnetic field lines, that is stretched from length ℓ_1 to ℓ_2. Volume V is conserved $\Rightarrow$ surface $S=\pi D^2 \sim \ell^{-1}$. From flux conservation $B\cdot S$=const it follows $B \sim \ell$. The magnetic energy in the cylinder $E_{mag} = B^2V/(2\mu_o)$ increases. In order to increase energy, work must be done against Lorentz forces $\mathbf{F}_L = \mathbf{j} \times \mathbf{B}$.



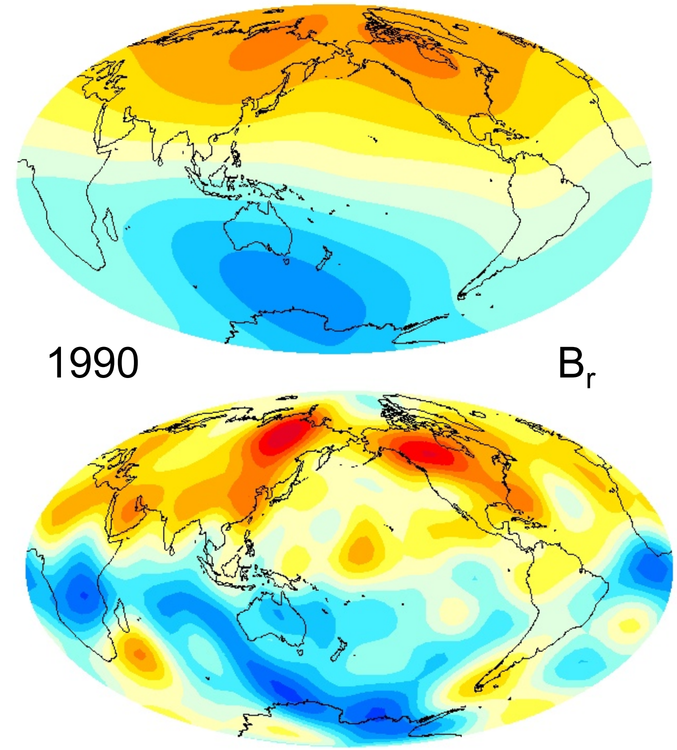
Stretching the fluid along field lines generates magnetic energy by doing work against Lorentz forces.

In a shear flow, the magnetic field vector is rotated and intensified. Differential rotation in a sphere is a special type of shear flow that generates axisymmetric toroidal field from an axial poloidal field. This is called the **ω-effect.**



Flow at the top of the Earth‘s core

At the top of Earth‘s core (more precisely below the Ekman layer at the top of the „free stream“) the flow is purely horizontal. Neglecting diffusion and considering the radial component of the induction equation we obtain (for $\nabla\cdot\mathbf{u}=0$) after some transformation:

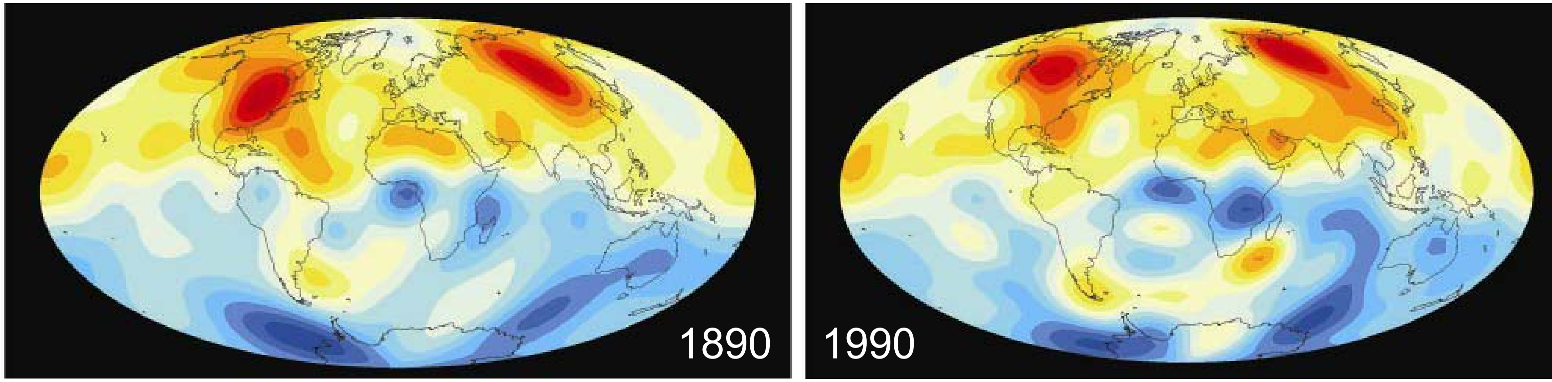


$$\partial B_r/\partial t = \nabla_h \cdot (B_r\mathbf{u}_h)$$

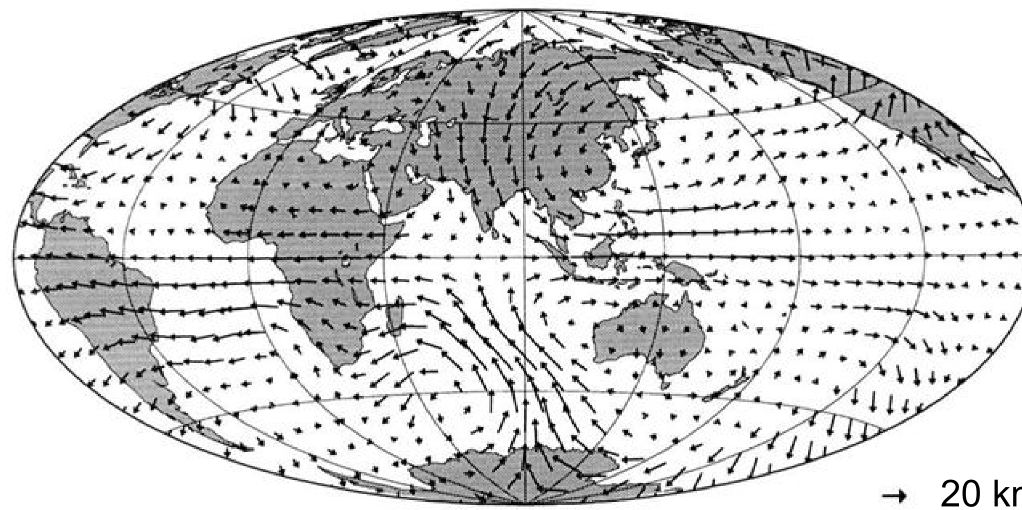
The magnetic field observed at the Earth’s surface can be downward continued to the top of the core, assuming that there are no sources of the field in the crust and mantle and using the expansion in spherical harmonics. Doing this at different times, we know both B_r and $\partial B_r/\partial t$ at the CMB. The task is to invert the equation and solve for $\mathbf{u}_h$. Because we have two unknown components of $\mathbf{u}_h$, but only one equation, additional assumptions are needed. Several have been employed: (1) Tangentially geostrophic flow, i.e. $\mathbf{u}_h$ is controlled by a balance of pressure and Coriolis force $\nabla_h\cdot(\cos\theta\ \mathbf{u}_h) = 0$. (2) Purely toroidal flow $\nabla_h\cdot\mathbf{u}_h = 0$. A smoothness assumption is made (flow complexity is penalized in the inversion).

Symbols: index h refers to the horizontal part

Flow at the top of the Earth's core



Example for a core-flow inversion. Robust results, independent of assumptions, are a typical flow magnitude ≈ 15 km/yr = 0.5 mm/s, westward flow in the tropical Atlantic and a vortex motion in the Indian ocean.



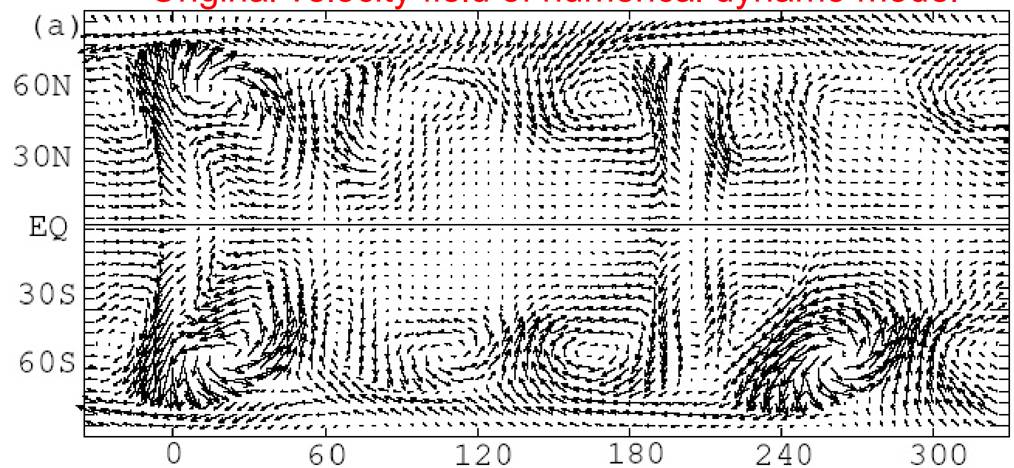
With this velocity estimate, the magnetic Reynolds number of the core is $Rm = UD/\lambda \approx 1000$. This is comfortably high for a dynamo, but not so high that it would not allow direct numerical simulation of the induction process.

Test of core flow inversion I

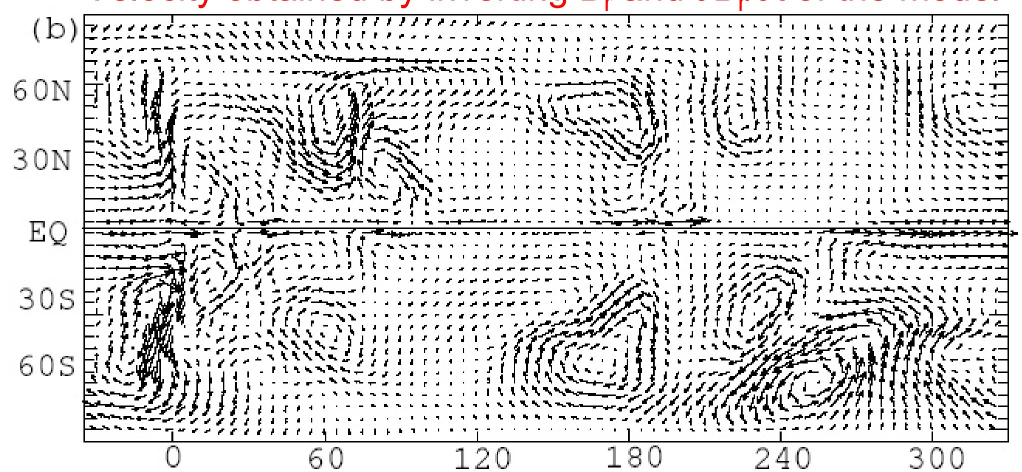
Several assumptions of uncertain validity enter into the core flow inversion, in particular the frozen-flux assumptions (negligible diffusion). How reliable is the inversion?

Tests using the numerical dynamo models suggest that the velocity is in the right order of magnitude and the flow pattern is matched in some regions but not in others.

Original velocity field of numerical dynamo model

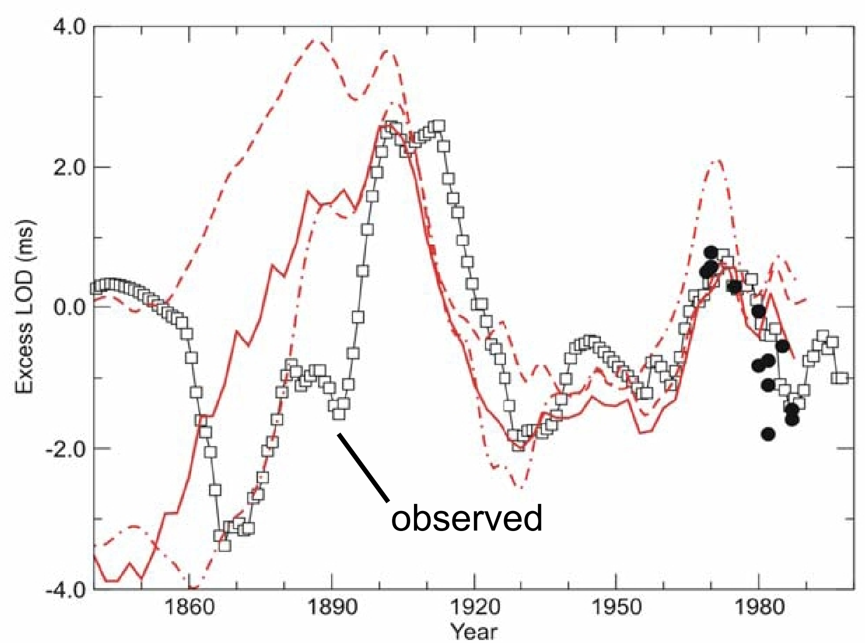


Velocity obtained by inverting B_r and $\partial B_r/\partial t$ of the model



Test of core flow inversion II

If the flow in the core changes with time, this may imply changes in the total angular momentum of the core. Since the angular momentum of the Earth as a whole must be conserved, a change in the core must be balanced by an opposite change of the mantle and crust contribution. At the Earth‘s surface this will be observed as a change of the rotation rate, or the **length of the day (LOD).** On time-scales of decades, LOD-changes of several msec are observed, for which no other explanation than angular momentum exchange with the core is viable.



Assume that the zonal (φ-independent) and equatorially symmetric components of u_φ obtained in a core-flow inversion represents (differential) flow on geostrophic cylinders cutting through the fluid core (no other flow components contribute to the angular momentum). The angular momentum change can be calculated from core-flow models at different epochs. The agreement with the observed LOD changes since 1900 is suprisingly good.