

回転球面上での 2 次元順圧流体の定式化

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この文書では回転 2 次元球面上の非圧縮流体の支配方程式と渦度方程式を導出することを行う。

導出の計算の見通しは 3 次元系において渦度方程式をベクトル形式で表現し、その後球面への拘束条件を適用し動径成分を書き下すやり方がお薦めである (理論マニュアル線形波動「ロスビー波 (2 次元非発散球面)」) が、以下では緯度経度動径座標の各成分の式をまず書き下し、式変形していくという手間のかかる計算を敢えて行う。

1 回転 2 次元球面上の非圧縮流体の支配方程式の導出

まず 3 次元回転系の非圧縮流体の支配方程式から回転 2 次元球面上の非圧縮流体の支配方程式を導出する。3 次元回転系での非圧縮流体の支配方程式を緯度経度動径座標 (λ, φ, r) で書き表すと、

$$\begin{aligned} \frac{\partial u}{\partial t} + \frac{u}{r \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{v}{r} \frac{\partial u}{\partial \varphi} + w \frac{\partial u}{\partial r} + \frac{uw - uv \tan \varphi}{r} \\ - 2\Omega \sin \varphi v + 2\Omega \cos \varphi w = - \frac{1}{\rho r \cos \varphi} \frac{\partial p}{\partial \lambda} \\ + \nu \left[\frac{1}{r^2 \cos^2 \varphi} \frac{\partial^2 u}{\partial \lambda^2} + \frac{1}{r^2} \frac{\partial^2 u}{\partial \varphi^2} - \frac{\tan \varphi}{r^2} \frac{\partial u}{\partial \varphi} + \frac{1}{r} \frac{\partial^2}{\partial r^2} (ru) \right. \\ \left. + \frac{2}{r^2 \cos \varphi} \frac{\partial w}{\partial \lambda} - \frac{2 \sin \varphi}{r^2 \cos^2 \varphi} \frac{\partial v}{\partial \lambda} - \frac{u}{r^2 \cos^2 \varphi} \right], \end{aligned} \quad (1)$$

$$\begin{aligned} \frac{\partial v}{\partial t} + \frac{u}{r \cos \varphi} \frac{\partial v}{\partial \lambda} + \frac{v}{r} \frac{\partial v}{\partial \varphi} + w \frac{\partial v}{\partial r} + \frac{vw + u^2 \tan \varphi}{r} + 2\Omega \sin \varphi u = -\frac{1}{\rho r} \frac{\partial p}{\partial \varphi} \\ + \nu \left[\frac{1}{r^2 \cos^2 \varphi} \frac{\partial^2 v}{\partial \lambda^2} + \frac{1}{r^2} \frac{\partial^2 v}{\partial \varphi^2} - \frac{\tan \varphi}{r^2} \frac{\partial v}{\partial \varphi} + \frac{1}{r} \frac{\partial^2}{\partial r^2} (rv) \right. \\ \left. + \frac{2}{r^2} \frac{\partial w}{\partial \varphi} + \frac{2 \sin \varphi}{r^2 \cos^2 \varphi} \frac{\partial u}{\partial \lambda} - \frac{v}{r^2 \cos^2 \varphi} \right], \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{\partial w}{\partial t} + \frac{u}{r \cos \varphi} \frac{\partial w}{\partial \lambda} + \frac{v}{r} \frac{\partial w}{\partial \varphi} + w \frac{\partial w}{\partial r} - \frac{u^2 + v^2}{r} - 2\Omega \cos \varphi u = -\frac{1}{\rho} \frac{\partial p}{\partial r} \\ + \nu \left[\frac{1}{r^2 \cos^2 \varphi} \frac{\partial^2 w}{\partial \lambda^2} + \frac{1}{r^2} \frac{\partial^2 w}{\partial \varphi^2} - \frac{\tan \varphi}{r^2} \frac{\partial w}{\partial \varphi} + \frac{1}{r} \frac{\partial^2}{\partial r^2} (rw) \right. \\ \left. - \frac{2}{r^2} \frac{\partial v}{\partial \varphi} + \frac{2v \tan \varphi}{r^2} - \frac{2}{r^2 \cos \varphi} \frac{\partial u}{\partial \lambda} - \frac{2w}{r^2} \right], \end{aligned} \quad (3)$$

$$\frac{1}{r \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{1}{r \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) + \frac{1}{r^2} \frac{\partial}{\partial r} (r^2 w) = 0. \quad (4)$$

ここで u, v, w はそれぞれ速度の緯度, 経度, 動径成分である. Ω は系の回転角速度でありその回転軸は北極向きである. 球面への拘束条件として, $u, v \propto r, w = 0$ を適用し, $r \rightarrow a$ の極限をとると, w のついた項は消え, 粘性項の $\frac{1}{r} \frac{\partial^2}{\partial r^2} (ru) = \frac{2u}{a^2}$, $\frac{1}{r} \frac{\partial^2}{\partial r^2} (rv) = \frac{2v}{a^2}$ となるので,

$$\frac{\partial u}{\partial t} + \frac{u}{a \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{v}{a} \frac{\partial u}{\partial \varphi} - \frac{uv \tan \varphi}{a} - 2\Omega \sin \varphi v = -\frac{1}{\rho a \cos \varphi} \frac{\partial p}{\partial \lambda} + f_\lambda, \quad (5)$$

$$\frac{\partial v}{\partial t} + \frac{u}{a \cos \varphi} \frac{\partial v}{\partial \lambda} + \frac{v}{a} \frac{\partial v}{\partial \varphi} + \frac{u^2 \tan \varphi}{a} + 2\Omega \sin \varphi u = -\frac{1}{\rho a} \frac{\partial p}{\partial \varphi} + f_\varphi, \quad (6)$$

$$\frac{1}{a \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) = 0. \quad (7)$$

ただし f_λ, f_φ は粘性項であり,

$$f_\lambda = \nu \left[\frac{1}{a^2 \cos^2 \varphi} \frac{\partial^2 u}{\partial \lambda^2} + \frac{1}{a^2} \frac{\partial^2 u}{\partial \varphi^2} - \frac{\tan \varphi}{a^2} \frac{\partial u}{\partial \varphi} + \frac{2u}{a^2} - \frac{2 \sin \varphi}{a^2 \cos^2 \varphi} \frac{\partial v}{\partial \lambda} - \frac{u}{a^2 \cos^2 \varphi} \right] \quad (8)$$

$$f_\varphi = \nu \left[\frac{1}{a^2 \cos^2 \varphi} \frac{\partial^2 v}{\partial \lambda^2} + \frac{1}{a^2} \frac{\partial^2 v}{\partial \varphi^2} - \frac{\tan \varphi}{a^2} \frac{\partial v}{\partial \varphi} + \frac{2v}{a^2} + \frac{2 \sin \varphi}{a^2 \cos^2 \varphi} \frac{\partial u}{\partial \lambda} - \frac{v}{a^2 \cos^2 \varphi} \right] \quad (9)$$

球面への拘束条件を適用した $u, v \propto r, w = 0$ は r が有限の元では正しい解にはならないことに注意されたい. 運動方程式の動径成分とつじつまをあわせることができなくなる. $r \rightarrow a$ の極限では動径方向には壁からの応力により球面へ運動が拘束されていると考えて運動方程式の動径成分は扱わない.

2 回転 2 次元球面上の非圧縮流体の渦度方程式の導出

ここでは回転する 2 次元球面上の非圧縮流体の支配方程式から渦度方程式を導出する。支配方程式は回転系での 2 次元球面上の運動方程式および連続の式 (5),(6),(7) である。渦度方程式は (5) に $-\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \cos \varphi$, (6) に $\frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda}$ を作用させることにより求まる。両式の時間変化項からは

$$\begin{aligned} & -\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \cos \varphi \frac{\partial u}{\partial t} + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} \frac{\partial v}{\partial t} = \frac{\partial}{\partial t} \left[-\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) + \frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \right] \\ & = \frac{\partial \zeta}{\partial t} \end{aligned}$$

ここで $\zeta = -\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) + \frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda}$ は渦度 (の動径成分) である。

非線形項からは

$$\begin{aligned} & -\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \cos \varphi \left[\frac{u}{a \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{v}{a} \frac{\partial u}{\partial \varphi} - \frac{uv \tan \varphi}{a} \right] \\ & + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} \left[\frac{u}{a \cos \varphi} \frac{\partial v}{\partial \lambda} + \frac{v}{a} \frac{\partial v}{\partial \varphi} + \frac{u^2 \tan \varphi}{a} \right] \end{aligned}$$

渦度の移流の形と連続の式の形を意識して変形していく。第 1 項目は

$$\begin{aligned} & -\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \left(\cos \varphi \frac{u}{a \cos \varphi} \frac{\partial u}{\partial \lambda} \right) = -\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \left(\frac{u}{a} \frac{\partial u}{\partial \lambda} \right) \\ & = -\frac{1}{a \cos \varphi} \frac{u}{a} \frac{\partial}{\partial \lambda} \left(\frac{\partial u}{\partial \varphi} \right) - \frac{1}{a^2 \cos \varphi} \left(\frac{\partial u}{\partial \lambda} \frac{\partial u}{\partial \varphi} \right) \\ & = -\frac{u}{a \cos \varphi} \frac{1}{a} \frac{\partial}{\partial \lambda} \left[\frac{\partial}{\partial \varphi} \frac{u \cos \varphi}{\cos \varphi} \right] - \frac{1}{a^2 \cos \varphi} \frac{\partial u}{\partial \lambda} \frac{\partial u}{\partial \varphi} \\ & = -\frac{u}{a \cos \varphi} \frac{1}{a} \frac{\partial}{\partial \lambda} \left[\frac{1}{\cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) + \frac{u \cos \varphi \sin \varphi}{\cos^2 \varphi} \right] - \frac{1}{a^2 \cos \varphi} \frac{\partial u}{\partial \lambda} \frac{\partial u}{\partial \varphi} \\ & = -\frac{u}{a \cos \varphi} \frac{\partial}{\partial \lambda} \left[\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] - \frac{u}{a \cos \varphi} \frac{\tan \varphi}{a} \frac{\partial u}{\partial \lambda} - \frac{1}{a^2 \cos \varphi} \frac{\partial u}{\partial \lambda} \frac{\partial u}{\partial \varphi} \end{aligned}$$

第 2 項目は

$$\begin{aligned} & -\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \left(\cos \varphi \frac{v}{a} \frac{\partial u}{\partial \varphi} \right) = -\frac{v}{a} \frac{\partial}{\partial \varphi} \left(\frac{1}{a} \frac{\partial u}{\partial \varphi} \right) - \frac{1}{a} \frac{\partial u}{\partial \varphi} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) \\ & = -\frac{v}{a} \frac{\partial}{\partial \varphi} \left[\frac{1}{a} \frac{\partial}{\partial \varphi} \left(\frac{u \cos \varphi}{\cos \varphi} \right) \right] - \frac{1}{a} \frac{\partial u}{\partial \varphi} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) \\ & = -\frac{v}{a} \frac{\partial}{\partial \varphi} \left[\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) + \frac{u \cos \varphi - \sin \varphi}{a} \right] - \frac{1}{a} \frac{\partial u}{\partial \varphi} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) \end{aligned}$$

$$= -\frac{v}{a} \frac{\partial}{\partial \varphi} \left[\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] - \frac{v}{a} \frac{\partial}{\partial \varphi} \left[\frac{u \tan \varphi}{a} \right] - \frac{1}{a} \frac{\partial u}{\partial \varphi} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi)$$

第 3 項目は

$$\begin{aligned} -\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \cos \varphi \left(-\frac{uv \tan \varphi}{a} \right) &= \frac{u \tan \varphi}{a} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) + \frac{v \cos \varphi}{a \cos \varphi} \frac{\partial}{\partial \varphi} \left(\frac{u \tan \varphi}{a} \right) \\ &= \frac{u \tan \varphi}{a} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) + \frac{v}{a} \frac{\partial}{\partial \varphi} \left(\frac{u \tan \varphi}{a} \right) \end{aligned}$$

第 4 項目は

$$\frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} \left(\frac{u}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \right) = \frac{u}{a \cos \varphi} \frac{\partial}{\partial \lambda} \left(\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \right) + \frac{1}{a^2 \cos^2 \varphi} \frac{\partial u}{\partial \lambda} \frac{\partial v}{\partial \lambda}$$

第 5 項目は

$$\begin{aligned} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} \left(\frac{v}{a} \frac{\partial v}{\partial \varphi} \right) &= \frac{v}{a} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \left(\frac{\partial v}{\partial \lambda} \right) + \frac{1}{a^2 \cos \varphi} \frac{\partial v}{\partial \lambda} \frac{\partial v}{\partial \varphi} \\ &= \frac{v}{a} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \left(\frac{\cos \varphi}{\cos \varphi} \frac{\partial v}{\partial \lambda} \right) + \frac{1}{a^2 \cos \varphi} \frac{\partial v}{\partial \lambda} \frac{\partial v}{\partial \varphi} \\ &= \frac{v}{a} \frac{1}{a} \frac{\partial}{\partial \varphi} \left(\frac{1}{\cos \varphi} \frac{\partial v}{\partial \lambda} \right) - \frac{v}{a} \frac{1}{a \cos \varphi} \left(\frac{\sin \varphi}{\cos \varphi} \frac{\partial v}{\partial \lambda} \right) + \frac{1}{a^2 \cos \varphi} \frac{\partial v}{\partial \lambda} \frac{\partial v}{\partial \varphi} \\ &= \frac{v}{a} \frac{\partial}{\partial \varphi} \left(\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \right) - \frac{v \tan \varphi}{a^2 \cos \varphi} \frac{\partial v}{\partial \lambda} + \frac{1}{a^2 \cos \varphi} \frac{\partial v}{\partial \lambda} \frac{\partial v}{\partial \varphi} \\ &= \frac{v}{a} \frac{\partial}{\partial \varphi} \left(\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \right) + \frac{1}{a^2 \cos \varphi} \frac{\partial v}{\partial \lambda} \left(\frac{\partial v}{\partial \varphi} - v \tan \varphi \right) \\ &= \frac{v}{a} \frac{\partial}{\partial \varphi} \left(\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \right) + \frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) \end{aligned}$$

第 6 項目は

$$\frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} \left(\frac{u^2 \tan \varphi}{a} \right) = \frac{u \tan \varphi}{a} \frac{1}{a \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{u \tan \varphi}{a} \frac{1}{a \cos \varphi} \frac{\partial u}{\partial \lambda}$$

全ての項を足しあわせると,

$$\begin{aligned} &= -\frac{u}{a \cos \varphi} \frac{\partial}{\partial \lambda} \left[\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] - \frac{u}{a \cos \varphi} \frac{\tan \varphi}{a} \frac{\partial u}{\partial \lambda} - \frac{1}{a^2 \cos \varphi} \frac{\partial u}{\partial \lambda} \frac{\partial u}{\partial \varphi} \\ &\quad - \frac{v}{a} \frac{\partial}{\partial \varphi} \left[\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] - \frac{v}{a} \frac{\partial}{\partial \varphi} \left[\frac{u \tan \varphi}{a} \right] - \frac{1}{a} \frac{\partial u}{\partial \varphi} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) \\ &\quad + \frac{u \tan \varphi}{a} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) + \frac{v}{a} \frac{\partial}{\partial \varphi} \left(\frac{u \tan \varphi}{a} \right) \\ &\quad + \frac{u}{a \cos \varphi} \frac{\partial}{\partial \lambda} \left(\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \right) + \frac{1}{a^2 \cos^2 \varphi} \frac{\partial u}{\partial \lambda} \frac{\partial v}{\partial \lambda} \end{aligned}$$

$$\begin{aligned}
 & + \frac{v}{a} \frac{\partial}{\partial \varphi} \left(\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \right) + \frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) \\
 & + \frac{u \tan \varphi}{a} \frac{1}{a \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{u \tan \varphi}{a} \frac{1}{a \cos \varphi} \frac{\partial u}{\partial \lambda} \\
 = & - \frac{u}{a \cos \varphi} \frac{\partial}{\partial \lambda} \left[\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] - \frac{1}{a^2 \cos \varphi} \frac{\partial u}{\partial \lambda} \frac{\partial u}{\partial \varphi} \\
 & - \frac{v}{a} \frac{\partial}{\partial \varphi} \left[\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] - \frac{1}{a} \frac{\partial u}{\partial \varphi} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) \\
 & + \frac{u \tan \varphi}{a} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) \\
 & + \frac{u}{a \cos \varphi} \frac{\partial}{\partial \lambda} \left(\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \right) + \frac{1}{a^2 \cos^2 \varphi} \frac{\partial u}{\partial \lambda} \frac{\partial v}{\partial \lambda} \\
 & + \frac{v}{a} \frac{\partial}{\partial \varphi} \left(\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \right) + \frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) \\
 & + \frac{u \tan \varphi}{a} \frac{1}{a \cos \varphi} \frac{\partial u}{\partial \lambda} \\
 = & \frac{u}{a \cos \varphi} \frac{\partial}{\partial \lambda} \left[\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} - \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] + \frac{v}{a} \frac{\partial}{\partial \varphi} \left[\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} - \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] \\
 & + \frac{u \tan \varphi}{a} \left[\frac{1}{a \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) \right] - \frac{1}{a} \frac{\partial u}{\partial \varphi} \left[\frac{1}{a \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) \right] \\
 & + \frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \left[\frac{1}{a \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) \right] \\
 = & \left[\frac{u}{a \cos \varphi} \frac{\partial}{\partial \lambda} + \frac{v}{a} \frac{\partial}{\partial \varphi} \right] \left[\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} - \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] \\
 & + \left[\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} - \frac{1}{a} \frac{\partial u}{\partial \varphi} + \frac{u \tan \varphi}{a} \right] \left[\frac{1}{a \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) \right] \\
 = & \left[\frac{u}{a \cos \varphi} \frac{\partial}{\partial \lambda} + \frac{v}{a} \frac{\partial}{\partial \varphi} \right] \left[\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} - \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] \\
 & + \left[\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} - \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] \left[\frac{1}{a \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) \right] \\
 = & \left[\frac{u}{a \cos \varphi} \frac{\partial}{\partial \lambda} + \frac{v}{a} \frac{\partial}{\partial \varphi} \right] \zeta + \zeta \left[\frac{1}{a \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) \right] \\
 = & \left[\frac{u}{a \cos \varphi} \frac{\partial}{\partial \lambda} + \frac{v}{a} \frac{\partial}{\partial \varphi} \right] \zeta
 \end{aligned}$$

最後に連続の式 (7) を用いた.

コリオリ力の項からは

$$- \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \cos \varphi (-2\Omega \sin \varphi v) + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} (2\Omega \sin \varphi u)$$

$$\begin{aligned}
 &= 2\Omega \frac{\sin \varphi}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) + 2\Omega \frac{v \cos \varphi}{a \cos \varphi} \frac{\partial}{\partial \varphi} (\sin \varphi) + 2\Omega \frac{\sin \varphi}{a \cos \varphi} \frac{\partial u}{\partial \lambda} \\
 &= 2\Omega \sin \varphi \left[\frac{1}{a \cos \varphi} \frac{\partial u}{\partial \lambda} + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) \right] + 2\Omega \frac{v}{a} \cos \varphi = 2\Omega \frac{v}{a} \cos \varphi.
 \end{aligned}$$

圧力項は

$$\begin{aligned}
 &-\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \cos \varphi \left(\frac{1}{\rho a \cos \varphi} \frac{\partial p}{\partial \lambda} \right) + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} \left(\frac{1}{\rho a} \frac{\partial p}{\partial \varphi} \right) \\
 &= -\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \left(\frac{1}{\rho a} \frac{\partial p}{\partial \lambda} \right) + \frac{1}{\rho a^2 \cos \varphi} \frac{\partial^2 p}{\partial \lambda \partial \varphi} \\
 &= -\frac{1}{\rho a^2 \cos \varphi} \frac{\partial^2 p}{\partial \lambda \partial \varphi} + \frac{1}{\rho a^2 \cos \varphi} \frac{\partial^2 p}{\partial \lambda \partial \varphi} = 0.
 \end{aligned}$$

粘性項は、まず f_λ の 2,3,6 項目をまとめて

$$\begin{aligned}
 &\frac{1}{a^2} \frac{\partial^2 u}{\partial \varphi^2} - \frac{\tan \varphi}{a^2} \frac{\partial u}{\partial \varphi} - \frac{u}{a^2 \cos^2 \varphi} = \frac{1}{a^2} \frac{\partial^2 u}{\partial \varphi^2} - \frac{1}{a^2} \frac{\partial}{\partial \varphi} (u \tan \varphi) \\
 &= \frac{1}{a^2} \frac{\partial}{\partial \varphi} \left(\frac{\partial u}{\partial \varphi} - u \tan \varphi \right) = \frac{1}{a^2} \frac{\partial}{\partial \varphi} \left[\frac{1}{\cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right]
 \end{aligned}$$

f_φ も同様に、

$$\frac{1}{a^2} \frac{\partial^2 v}{\partial \varphi^2} - \frac{\tan \varphi}{a^2} \frac{\partial v}{\partial \varphi} - \frac{v}{a^2 \cos^2 \varphi} = \frac{1}{a^2} \frac{\partial}{\partial \varphi} \left[\frac{1}{\cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) \right]$$

したがって、

$$\begin{aligned}
 &-\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (\cos \varphi f_\lambda) + \frac{1}{a \cos \varphi} \frac{\partial f_\varphi}{\partial \lambda} \\
 &= -\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \cos \varphi \left\{ \frac{1}{a^2 \cos^2 \varphi} \frac{\partial^2 u}{\partial \lambda^2} + \frac{1}{a^2} \frac{\partial}{\partial \varphi} \left[\frac{1}{\cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] + \frac{2u}{a^2} - \frac{2 \sin \varphi}{a^2 \cos^2 \varphi} \frac{\partial v}{\partial \lambda} \right\} \\
 &\quad + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} \left\{ \frac{1}{a^2 \cos^2 \varphi} \frac{\partial^2 v}{\partial \lambda^2} + \frac{1}{a^2} \frac{\partial}{\partial \varphi} \left[\frac{1}{\cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) \right] + \frac{2v}{a^2} + \frac{2 \sin \varphi}{a^2 \cos^2 \varphi} \frac{\partial u}{\partial \lambda} \right\}
 \end{aligned}$$

これより渦度の形を意識しながら変形していく。第 1 項目は

$$\begin{aligned}
 &-\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \cos \varphi \left\{ \frac{1}{a^2 \cos^2 \varphi} \frac{\partial^2 u}{\partial \lambda^2} \right\} = -\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \left[\frac{1}{a^2 \cos \varphi} \frac{\partial^2 u}{\partial \lambda^2} \right] \\
 &= -\frac{1}{a \cos \varphi} \frac{\partial^2}{\partial \lambda^2} \frac{\partial}{\partial \varphi} \left(\frac{u}{a^2 \cos \varphi} \right) = -\frac{1}{a \cos \varphi} \frac{\partial^2}{\partial \lambda^2} \frac{\partial}{\partial \varphi} \left(\frac{u \cos \varphi}{a^2 \cos^2 \varphi} \right) \\
 &= -\frac{1}{a \cos \varphi} \frac{\partial^2}{\partial \lambda^2} \left[\frac{1}{a^2 \cos^2 \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) + u \cos \varphi \frac{2 \sin \varphi \cos \varphi}{a^2 \cos^4 \varphi} \right] \\
 &= -\frac{1}{a^2 \cos^2 \varphi} \frac{\partial^2}{\partial \lambda^2} \left[\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] - \frac{1}{a \cos \varphi} \frac{\partial^2}{\partial \lambda^2} \left(\frac{2u \sin \varphi}{a^2 \cos^2 \varphi} \right) \\
 &= -\frac{1}{a^2 \cos^2 \varphi} \frac{\partial^2}{\partial \lambda^2} \left[\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] - \frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} \left(\frac{2 \sin \varphi}{a^2 \cos^2 \varphi} \frac{\partial u}{\partial \lambda} \right).
 \end{aligned}$$

第 2 項目は

$$\begin{aligned}
 & -\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \cos \varphi \left\{ \frac{1}{a^2} \frac{\partial}{\partial \varphi} \left[\frac{1}{\cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] \right\} \\
 & = -\frac{1}{a^2 \cos \varphi} \frac{\partial}{\partial \varphi} \left\{ \cos \varphi \frac{\partial}{\partial \varphi} \left[\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] \right\}.
 \end{aligned}$$

第 3 項目は

$$-\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \left(\cos \varphi \frac{2u}{a^2} \right) = -\frac{2}{a^2} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi).$$

第 4 項目は

$$-\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \cos \varphi \left(-\frac{2 \sin \varphi}{a^2 \cos^2 \varphi} \frac{\partial v}{\partial \lambda} \right) = \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \left(\frac{2 \sin \varphi}{a^2 \cos \varphi} \frac{\partial v}{\partial \lambda} \right).$$

第 5 項目は

$$\frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} \left[\frac{1}{a^2 \cos^2 \varphi} \frac{\partial^2 v}{\partial \lambda^2} \right] = \frac{1}{a^2 \cos^2 \varphi} \frac{\partial^2}{\partial \lambda^2} \left[\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \right].$$

第 6 項目は

$$\begin{aligned}
 & \frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} \left\{ \frac{1}{a^2} \frac{\partial}{\partial \varphi} \left[\frac{1}{\cos \varphi} \frac{\partial}{\partial \varphi} (v \cos \varphi) \right] \right\} = \frac{1}{a^3 \cos \varphi} \frac{\partial}{\partial \varphi} \left[\frac{1}{\cos \varphi} \frac{\partial}{\partial \varphi} \left(\cos \varphi \frac{\partial v}{\partial \lambda} \right) \right] \\
 & = \frac{1}{a^3 \cos \varphi} \frac{\partial}{\partial \varphi} \left[\frac{1}{\cos \varphi} \frac{\partial}{\partial \varphi} \left(\frac{\cos \varphi}{\cos \varphi^2} \frac{\partial v}{\partial \lambda} \right) \right] \\
 & = \frac{1}{a^3 \cos \varphi} \frac{\partial}{\partial \varphi} \left[\frac{1}{\cos \varphi} \left(\cos^2 \varphi \frac{\partial}{\partial \varphi} \left(\frac{1}{\cos \varphi} \frac{\partial v}{\partial \lambda} \right) + \frac{1}{\cos \varphi} \frac{\partial v}{\partial \lambda} (-2 \sin \varphi \cos \varphi) \right) \right] \\
 & = \frac{1}{a^3 \cos \varphi} \frac{\partial}{\partial \varphi} \left[\cos \varphi \frac{\partial}{\partial \varphi} \left(\frac{1}{\cos \varphi} \frac{\partial v}{\partial \lambda} \right) - \frac{2 \sin \varphi}{\cos \varphi} \frac{\partial v}{\partial \lambda} \right] \\
 & = \frac{1}{a^2 \cos \varphi} \frac{\partial}{\partial \varphi} \left[\cos \varphi \frac{\partial}{\partial \varphi} \left(\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \right) \right] - \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \left(\frac{2 \sin \varphi}{a^2 \cos \varphi} \frac{\partial v}{\partial \lambda} \right).
 \end{aligned}$$

第 7 項目は

$$\frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} \left(\frac{2v}{a^2} \right) = \frac{2}{a^2} \frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda}.$$

第 8 項目はそのまま

$$\frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} \left(\frac{2 \sin \varphi}{a^2 \cos^2 \varphi} \frac{\partial u}{\partial \lambda} \right).$$

全てまとめると

$$\begin{aligned}
 & -\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (\cos \varphi f_\lambda) + \frac{1}{a \cos \varphi} \frac{\partial f_\varphi}{\partial \lambda} \\
 = & -\frac{1}{a^2 \cos^2 \varphi} \frac{\partial^2}{\partial \lambda^2} \left[\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] - \frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} \left(\frac{2 \sin \varphi}{a^2 \cos^2 \varphi} \frac{\partial u}{\partial \lambda} \right) \\
 & -\frac{1}{a^2 \cos \varphi} \frac{\partial}{\partial \varphi} \left\{ \cos \varphi \frac{\partial}{\partial \varphi} \left[\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] \right\} \\
 & -\frac{2}{a^2} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \left(\frac{2 \sin \varphi}{a^2 \cos^2 \varphi} \frac{\partial v}{\partial \lambda} \right) \\
 & + \frac{1}{a^2 \cos^2 \varphi} \frac{\partial^2}{\partial \lambda^2} \left[\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \right] + \frac{1}{a^2 \cos \varphi} \frac{\partial}{\partial \varphi} \left[\cos \varphi \frac{\partial}{\partial \varphi} \left(\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \right) \right] \\
 & -\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} \left(\frac{2 \sin \varphi}{a^2 \cos^2 \varphi} \frac{\partial v}{\partial \lambda} \right) + \frac{2}{a^2} \frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} + \frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} \left(\frac{2 \sin \varphi}{a^2 \cos^2 \varphi} \frac{\partial u}{\partial \lambda} \right) \\
 = & -\frac{1}{a^2 \cos^2 \varphi} \frac{\partial^2}{\partial \lambda^2} \left[\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] \\
 & -\frac{1}{a^2 \cos \varphi} \frac{\partial}{\partial \varphi} \left\{ \cos \varphi \frac{\partial}{\partial \varphi} \left[\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] \right\} \\
 & -\frac{2}{a^2} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) + \frac{1}{a^2 \cos^2 \varphi} \frac{\partial^2}{\partial \lambda^2} \left[\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \right] \\
 & + \frac{1}{a^2 \cos \varphi} \frac{\partial}{\partial \varphi} \left[\cos \varphi \frac{\partial}{\partial \varphi} \left(\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \right) \right] + \frac{2}{a^2} \frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \\
 = & \frac{1}{a^2 \cos^2 \varphi} \frac{\partial^2}{\partial \lambda^2} \left[\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} - \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] \\
 & + \frac{1}{a^2 \cos \varphi} \frac{\partial}{\partial \varphi} \left[\cos \varphi \frac{\partial}{\partial \varphi} \left(\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} - \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right) \right] \\
 & + \frac{2}{a^2} \left[\frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} - \frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) \right] \\
 = & \frac{1}{a^2 \cos^2 \varphi} \frac{\partial^2 \zeta}{\partial \lambda^2} + \frac{1}{a^2 \cos \varphi} \frac{\partial}{\partial \varphi} \left(\cos \varphi \frac{\partial \zeta}{\partial \varphi} \right) + \frac{2\zeta}{a^2}.
 \end{aligned}$$

これより, 渦度方程式は

$$\begin{aligned}
 \frac{\partial \zeta}{\partial t} + \left[\frac{u}{a \cos \varphi} \frac{\partial}{\partial \lambda} + \frac{v}{a} \frac{\partial}{\partial \varphi} \right] \zeta + 2\Omega \frac{v}{a} \cos \varphi \\
 = \nu \left[\frac{1}{a^2 \cos^2 \varphi} \frac{\partial^2 \zeta}{\partial \lambda^2} + \frac{1}{a^2 \cos \varphi} \frac{\partial}{\partial \varphi} \left(\cos \varphi \frac{\partial \zeta}{\partial \varphi} \right) + \frac{2\zeta}{a^2} \right].
 \end{aligned}$$

ここで流線関数 ψ を

$$u = -\frac{1}{a} \frac{\partial \psi}{\partial \varphi}, \quad v = \frac{1}{a \cos \varphi} \frac{\partial \psi}{\partial \lambda}, \quad (10)$$

として定義すると、連続の式 (7) を自動的に満たす。渦度 ζ は

$$\begin{aligned}\zeta &= -\frac{1}{a \cos \varphi} \frac{\partial}{\partial \varphi} (u \cos \varphi) + \frac{1}{a \cos \varphi} \frac{\partial v}{\partial \lambda} \\ &= \frac{1}{a^2 \cos \varphi} \frac{\partial}{\partial \varphi} \left(\cos \varphi \frac{\partial \psi}{\partial \varphi} \right) + \frac{1}{a^2 \cos^2 \varphi} \frac{\partial^2 \psi}{\partial \lambda^2} = \nabla^2 \psi.\end{aligned}$$

ここで $\nabla^2 = \frac{1}{a^2 \cos^2 \varphi} \frac{\partial^2}{\partial \lambda^2} + \frac{1}{a^2 \cos \varphi} \frac{\partial}{\partial \varphi} \cos \varphi \frac{\partial}{\partial \varphi}$ は球面上の水平ラプラシアンである。これらを渦度方程式 (10) に代入すると、

$$\frac{\partial \nabla^2 \psi}{\partial t} + \left[-\frac{1}{a} \frac{\partial \psi}{\partial \varphi} \frac{1}{a \cos \varphi} \frac{\partial}{\partial \lambda} + \frac{1}{a \cos \varphi} \frac{\partial \psi}{\partial \lambda} \frac{1}{a} \frac{\partial}{\partial \varphi} \right] \nabla^2 \psi + \frac{2\Omega}{a^2} \frac{\partial \psi}{\partial \lambda} = \nu \left(\nabla^2 + \frac{2}{a^2} \right) \nabla^2 \psi.$$

さらに $\sin$ 緯度 $\mu = \sin \varphi$ を導入すると、 $\frac{\partial}{\partial \mu} = \frac{1}{\cos \varphi} \frac{\partial}{\partial \varphi}$ であるから、

$$\frac{\partial \nabla^2 \psi}{\partial t} + \left[-\frac{1}{a^2} \frac{\partial \psi}{\partial \mu} \frac{\partial}{\partial \lambda} + \frac{1}{a^2} \frac{\partial \psi}{\partial \lambda} \frac{\partial}{\partial \mu} \right] \nabla^2 \psi + \frac{2\Omega}{a^2} \frac{\partial \psi}{\partial \lambda} = \nu \left(\nabla^2 + \frac{2}{a^2} \right) \nabla^2 \psi.$$

よって

$$\frac{\partial \nabla^2 \psi}{\partial t} + \frac{1}{a^2} J(\psi, \nabla^2 \psi) + \frac{2\Omega}{a^2} \frac{\partial \psi}{\partial \lambda} = \nu \left(\nabla^2 + \frac{2}{a^2} \right) \nabla^2 \psi. \quad (11)$$

ただし $J(f, g) = (\partial_\lambda f)(\partial_\mu g) - (\partial_\lambda g)(\partial_\mu f)$ はヤコビアンである。ラプラシアンを μ で表しておく

$$\nabla^2 = \frac{1}{a^2 \cos^2 \varphi} \frac{\partial^2}{\partial \lambda^2} + \frac{1}{a^2 \cos \varphi} \frac{\partial}{\partial \varphi} \cos \varphi \frac{\partial}{\partial \varphi} = \frac{1}{a^2(1-\mu^2)} \frac{\partial^2}{\partial \lambda^2} + \frac{1}{a^2} \frac{\partial}{\partial \mu} (1-\mu^2) \frac{\partial}{\partial \mu}. \quad (12)$$

3 角運動量保存則

ψ の球面調和関数 $Y_n^m(\mu, \lambda)$ で展開した成分のうち、 $n=0$ は剛体回転流を表わす。 $(n, m) = (1, 0)$ は極方向を回転軸とする剛体回転流、 $(n, m) = (1, 1), (1, -1)$ は赤道面上の軸を回転軸とする剛体回転流である。したがってこれらの成分は各々の軸に対する角運動量となっており、非回転系では全て、回転系では $(n, m) = (1, 0)$ 成分だけ保存しているはずである。そのことを確かめてみよう。

いま流線関数が球面調和関数の足しあわせで

$$\psi(\lambda, \mu, t) = \sum_{n=1}^{\infty} \sum_{m=-n}^n \tilde{\psi}_{n,m} Y_n^m(\lambda, \mu) = \sum_{n=1}^{\infty} \sum_{m=-n}^n \tilde{\psi}_{n,m} P_n^m(\lambda) e^{i\mu\lambda} \quad (13)$$

と表わされているとする. このとき ψ が実数であることから係数には

$$\tilde{\psi}_{n,-m} = \tilde{\psi}_{n,m}^* \quad (14)$$

が成り立つ.

まず $(n, m) = (1, 0)$ 成分について考えよう. 渦度方程式 (11) において, 粘性項の 1,0 成分は

$$\left(\nabla^2 + \frac{2}{a^2} \right) \nabla^2 \tilde{\psi}_{1,0} Y_1^0(\lambda, \mu) = \left(\frac{-1 \times 2}{a^2} + \frac{2}{a^2} \right) \frac{-1 \times 2}{a^2} \tilde{\psi}_{1,0} Y_1^0(\lambda, \mu) = 0. \quad (15)$$

(11) の β 項の 1,0 成分は

$$\frac{2\Omega}{a^2} \frac{\partial}{\partial \lambda} \tilde{\psi}_{1,0} Y_1^0(\lambda, \mu) = 0. \quad (16)$$

非線形項からの寄与は, (13) をヤコビアンに代入して $Y_1^0(\lambda, \mu) = \mu$ をかけて球面上で積分することにより得られる.

$$\begin{aligned} & \int_0^{2\pi} \int_{-1}^1 d\lambda d\mu \frac{1}{a^2} J(\psi, \nabla^2 \psi) Y_1^0(\lambda, \mu) \\ &= \frac{1}{a^2} \sum_{n,m} \sum_{n',m'} \int_0^{2\pi} \int_{-1}^1 \left[im \tilde{\psi}_{n,m} \tilde{\psi}_{n',m'} \mu Y_n^m \frac{\partial Y_{n'}^{m'}}{\partial \mu} - im' \tilde{\psi}_{n,m} \tilde{\psi}_{n',m'} \mu \frac{\partial Y_n^m}{\partial \mu} Y_{n'}^{m'} \right] d\lambda d\mu \\ &= \frac{1}{a^2} \sum_{n,m} \sum_{n',m'} \tilde{\psi}_{n,m} \tilde{\psi}_{n',m'} \int_0^{2\pi} \int_{-1}^1 \left[im \mu Y_n^m \frac{\partial Y_{n'}^{m'}}{\partial \mu} - im' \mu \frac{\partial Y_n^m}{\partial \mu} Y_{n'}^{m'} \right] d\lambda d\mu \end{aligned}$$

ここで $m = -m'$ 成分のみ 0 でないことに注意すると

$$\begin{aligned} & \int_0^{2\pi} \int_{-1}^1 d\lambda d\mu \left[im \mu Y_n^m \frac{\partial Y_{n'}^{m'}}{\partial \mu} - im' \mu \frac{\partial Y_n^m}{\partial \mu} Y_{n'}^{m'} \right] \\ &= \int_0^{2\pi} \left[im \mu Y_n^m Y_{n'}^{m'} \right]_{-1}^1 d\lambda - \int_0^{2\pi} \int_{-1}^1 im Y_n^m Y_{n'}^{m'} d\lambda d\mu - \int_0^{2\pi} \int_{-1}^1 im \mu \frac{\partial Y_n^m}{\partial \mu} Y_{n'}^{m'} d\lambda d\mu \\ & \quad + \int_0^{2\pi} \int_{-1}^1 \left[-im' \mu \frac{\partial Y_n^m}{\partial \mu} Y_{n'}^{m'} \right] d\lambda d\mu \\ &= \int_0^{2\pi} \left[im \mu Y_n^m Y_{n'}^{-m} \right]_{-1}^1 d\lambda - \int_0^{2\pi} \int_{-1}^1 im Y_n^m Y_{n'}^{-m} d\lambda d\mu - \int_0^{2\pi} \int_{-1}^1 im \mu \frac{\partial Y_n^m}{\partial \mu} Y_{n'}^{-m} d\lambda d\mu \\ & \quad + \int_0^{2\pi} \int_{-1}^1 im \mu \frac{\partial Y_n^m}{\partial \mu} Y_{n'}^{-m} d\lambda d\mu \\ &= \int_0^{2\pi} \left[im \mu Y_n^m Y_{n'}^{-m} \right]_{-1}^1 d\lambda - \int_0^{2\pi} \int_{-1}^1 im Y_n^m Y_{n'}^{-m} d\lambda d\mu \\ &= \int_0^{2\pi} \left[im \mu Y_n^m Y_{n'}^{-m} \right]_{-1}^1 d\lambda - \int_0^{2\pi} \int_{-1}^1 im Y_n^m Y_{n'}^{-m} d\lambda d\mu \end{aligned}$$

最後に Y_n^m の直交関係を用いている. 第 1 項目の値を評価する. Y_n^m の $\mu = \pm 1$ の値は $m \neq 0$ のとき 0 である. $m = 0$ のときは係数 m がかかっているので結局この項は 0 となる. したがって, n, m, n', m' 成分からは

$$-\tilde{\psi}_{n,m}\tilde{\psi}_{n,-m}\int_0^{2\pi}\int_{-1}^1 imY_n^mY_n^{-m}d\lambda d\mu = -|\tilde{\psi}_{n,m}|^2\int_0^{2\pi}\int_{-1}^1 imY_n^mY_n^{-m}d\lambda d\mu \quad (17)$$

一方, $n, -m$ から

$$-|\tilde{\psi}_{n,-m}|^2\int_0^{2\pi}\int_{-1}^1 (-imY_n^{-m}Y_n^m)d\lambda d\mu = |\tilde{\psi}_{n,m}^*|^2\int_0^{2\pi}\int_{-1}^1 imY_n^{-m}Y_n^m d\lambda d\mu \quad (18)$$

したがってこの項は互いにキャンセルする. 結局非線形項からの $(n, m) = (1, 0)$ への寄与は 0 となる. 最後に時間変化項は

$$\frac{\partial}{\partial t}\nabla^2\tilde{\psi}_{1,0}(t)Y_1^0 = -\frac{1(1+1)}{a^2}\frac{d\tilde{\psi}_{1,0}}{dt}Y_1^0 \quad (19)$$

よって

$$\frac{d\tilde{\psi}_{1,0}}{dt} = 0 \quad (20)$$

となり極方向を軸とする角運動量は保存する.

次に $(n, m) = (1, 1)$ 成分について考えよう. 粘性項の $(1, 1)$ 成分は $(n, m) = (1, 0)$ と同様にして 0 である. (11) の β 項の $1, 1$ 成分は

$$\frac{2\Omega}{a^2}\frac{\partial}{\partial\lambda}\tilde{\psi}_{1,1}Y_1^1(\lambda, \mu) = i\frac{2\Omega}{a^2}\tilde{\psi}_{1,1}Y_1^1(\lambda, \mu). \quad (21)$$

非線形項からの寄与は, (13) をヤコビアンに代入して $Y_1^1(\lambda, \mu) = \sqrt{1-\mu^2}e^{i\lambda}$ をかけて球面上で積分することにより得られる.

$$\begin{aligned} & \int_0^{2\pi}\int_{-1}^1 d\lambda d\mu \frac{1}{a^2} J(\psi, \nabla^2\psi) Y_1^1(\lambda, \mu) \\ &= \frac{1}{a^2} \sum_{n,m} \sum_{n',m'} \int_0^{2\pi} \int_{-1}^1 \left[im\tilde{\psi}_{n,m}\tilde{\psi}_{n',m'}\sqrt{1-\mu^2}e^{i\lambda}Y_n^m\frac{\partial Y_{n'}^{m'}}{\partial\mu} \right. \\ & \quad \left. - im'\tilde{\psi}_{n,m}\tilde{\psi}_{n',m'}\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'} \right] d\lambda d\mu \\ &= \frac{1}{a^2} \sum_{n,m} \sum_{n',m'} \tilde{\psi}_{n,m}\tilde{\psi}_{n',m'} \int_0^{2\pi} \int_{-1}^1 \left[im\sqrt{1-\mu^2}e^{i\lambda}Y_n^m\frac{\partial Y_{n'}^{m'}}{\partial\mu} - im'\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'} \right] d\lambda d\mu \end{aligned}$$

ここで $m' = -1 - m$ 成分のみ 0 でないことに注意すると, $m = n$ のときは 0 になり, $0 < m < n + 1$ なる n, m に対して

$$\int_0^{2\pi} \int_{-1}^1 d\lambda d\mu \left[im\sqrt{1-\mu^2}e^{i\lambda}Y_n^m\frac{\partial Y_{n'}^{m'}}{\partial\mu} - im'\sqrt{1-\mu^2}e^{i\lambda}\frac{\partial Y_n^m}{\partial\mu}Y_{n'}^{m'} \right]$$

$$\begin{aligned}
&= \int_0^{2\pi} \left[im\sqrt{1-\mu^2}e^{i\lambda}Y_n^m Y_{n'}^{m'} \right]_{-1}^1 d\lambda - \int_0^{2\pi} \int_{-1}^1 -im \frac{\mu}{\sqrt{1-\mu^2}} e^{i\lambda} Y_n^m Y_{n'}^{m'} d\lambda d\mu \\
&\quad - \int_0^{2\pi} \int_{-1}^1 im\sqrt{1-\mu^2}e^{i\lambda} \frac{\partial Y_n^m}{\partial \mu} Y_{n'}^{m'} d\lambda d\mu + \int_0^{2\pi} \int_{-1}^1 \left[-im' \sqrt{1-\mu^2} e^{i\lambda} \frac{\partial Y_n^m}{\partial \mu} Y_{n'}^{m'} \right] d\lambda d\mu \\
&= \int_0^{2\pi} \int_{-1}^1 im \frac{\mu}{\sqrt{1-\mu^2}} e^{i\lambda} Y_n^m Y_{n'}^{m'} d\lambda d\mu - i(m+m') \int_0^{2\pi} \int_{-1}^1 \sqrt{1-\mu^2} e^{i\lambda} \frac{\partial Y_n^m}{\partial \mu} Y_{n'}^{m'} d\lambda d\mu \\
&= \int_0^{2\pi} \int_{-1}^1 im \frac{\mu}{\sqrt{1-\mu^2}} e^{i\lambda} Y_n^m Y_{n'}^{m'} d\lambda d\mu + i \int_0^{2\pi} \int_{-1}^1 \sqrt{1-\mu^2} e^{i\lambda} \frac{\partial Y_n^m}{\partial \mu} Y_{n'}^{m'} d\lambda d\mu \\
&= i \int_0^{2\pi} \int_{-1}^1 \frac{1}{\sqrt{1-\mu^2}} e^{i\lambda} \left[m\mu Y_n^m + (1-\mu^2) \frac{\partial Y_n^m}{\partial \mu} \right] Y_{n'}^{m'} d\lambda d\mu
\end{aligned}$$

ここでルジャンドル関数の漸化式

$$(1-\mu^2) \frac{\partial P_n^m}{\partial \mu} = \sqrt{1-\mu^2} P_n^{m+1} - m\mu P_n^m \quad (22)$$

より

$$\begin{aligned}
e^{i\lambda} \left[m\mu Y_n^m + (1-\mu^2) \frac{\partial Y_n^m}{\partial \mu} \right] &= e^{i(m+1)\lambda} \left[m\mu P_n^m + (1-\mu^2) \frac{\partial P_n^m}{\partial \mu} \right] \\
&= \sqrt{1-\mu^2} P_n^{m+1} e^{i(m+1)\lambda} = \sqrt{1-\mu^2} Y_n^{m+1}.
\end{aligned}$$

したがって, n, m 成分からの寄与は

$$\begin{aligned}
&\frac{1}{a^2} \tilde{\psi}_{n,m} \tilde{\psi}_{n',m'} \int_0^{2\pi} \int_{-1}^1 d\lambda d\mu \left[im\sqrt{1-\mu^2}e^{i\lambda}Y_n^m \frac{\partial Y_{n'}^{m'}}{\partial \mu} - im' \sqrt{1-\mu^2}e^{i\lambda} \frac{\partial Y_n^m}{\partial \mu} Y_{n'}^{m'} \right] \\
&= i \frac{1}{a^2} \tilde{\psi}_{n,m} \tilde{\psi}_{n',-m-1} \int_0^{2\pi} \int_{-1}^1 Y_n^{m+1} Y_{n'}^{m'} d\lambda d\mu d\lambda d\mu \\
&= i \frac{1}{a^2} \tilde{\psi}_{n,m} \tilde{\psi}_{n,-m-1} \int_0^{2\pi} \int_{-1}^1 Y_n^{m+1} Y_n^{-m-1} d\lambda d\mu d\lambda d\mu
\end{aligned}$$

最後に直交関係を使って $n' = n$ とした.

一方 $m < 0$ の成分に対して, $m \rightarrow -m-1 > -n-1$ を選んで計算すると $m' = m$ の時のみ 0 でなく,

$$\begin{aligned}
&\int_0^{2\pi} \int_{-1}^1 d\lambda d\mu \left[im\sqrt{1-\mu^2}e^{i\lambda}Y_n^m \frac{\partial Y_{n'}^{m'}}{\partial \mu} - im' \sqrt{1-\mu^2}e^{i\lambda} \frac{\partial Y_n^m}{\partial \mu} Y_{n'}^{m'} \right] \\
&= \int_0^{2\pi} \int_{-1}^1 d\lambda d\mu \left[i(-m-1)\sqrt{1-\mu^2}e^{i\lambda}Y_n^{-m-1} \frac{\partial Y_{n'}^{m'}}{\partial \mu} - im' \sqrt{1-\mu^2}e^{i\lambda} \frac{\partial Y_n^{-m-1}}{\partial \mu} Y_{n'}^{m'} \right] \\
&= \int_0^{2\pi} \int_{-1}^1 d\lambda d\mu \left[i(-m-1)\sqrt{1-\mu^2}e^{i\lambda}Y_n^{-m-1} \frac{\partial Y_{n'}^m}{\partial \mu} - im \sqrt{1-\mu^2}e^{i\lambda} \frac{\partial Y_n^{-m-1}}{\partial \mu} Y_{n'}^m \right]
\end{aligned}$$

$$\begin{aligned}
&= - \int_0^{2\pi} \int_{-1}^1 d\lambda d\mu \left[im\sqrt{1-\mu^2}e^{i\lambda}Y_n^m \frac{\partial Y_n^{m-1}}{\partial \mu} - i(-m-1)\sqrt{1-\mu^2}e^{i\lambda} \frac{\partial Y_n^m}{\partial \mu} Y_n^{m-1} \right] \\
&= -i \int_0^{2\pi} \int_{-1}^1 Y_n^{m+1} Y_n^{m-1} d\lambda d\mu d\lambda d\mu \\
&= -i \int_0^{2\pi} \int_{-1}^1 Y_n^{m+1} Y_n^{m-1} d\lambda d\mu d\lambda d\mu.
\end{aligned}$$

最後の計算は n, m の時の式変形と同様に行った. したがって, $n, -m-1$ 成分からの寄与は

$$\begin{aligned}
&\frac{1}{a^2} \tilde{\psi}_{n,-m-1} \tilde{\psi}_{n',m'} \int_0^{2\pi} \int_{-1}^1 d\lambda d\mu \left[im\sqrt{1-\mu^2}e^{i\lambda}Y_n^m \frac{\partial Y_n^{m'}}{\partial \mu} - im'\sqrt{1-\mu^2}e^{i\lambda} \frac{\partial Y_n^m}{\partial \mu} Y_n^{m'} \right] \\
&= -i \frac{1}{a^2} \tilde{\psi}_{n,-m-1} \tilde{\psi}_{n',m} \int_0^{2\pi} \int_{-1}^1 Y_n^{m-1} Y_n^m d\lambda d\mu d\lambda d\mu \\
&= -i \frac{1}{a^2} \tilde{\psi}_{n,-m-1} \tilde{\psi}_{n,m} \int_0^{2\pi} \int_{-1}^1 Y_n^{m-1} Y_n^m d\lambda d\mu d\lambda d\mu
\end{aligned}$$

よって n, m 成分と $n, -m-1$ 成分からの寄与が打ち消しあって非線形項は 0 となる.

最後に時間変化項は $(1, 0)$ 成分と同じであり

$$\frac{\partial}{\partial t} \nabla^2 \tilde{\psi}_{1,1}(t) Y_1^1 = -\frac{1(1+1)}{a^2} \frac{d\tilde{\psi}_{1,1}}{dt} Y_1^1 \quad (23)$$

よって $(1, 1)$ 成分の式は

$$-\frac{2}{a^2} \frac{d\tilde{\psi}_{1,1}}{dt} + i \frac{2\Omega}{a^2} \tilde{\psi}_{1,1} = 0, \quad \tilde{\psi}_1^1 = Ae^{i\Omega t}. \quad (24)$$

A は任意定数であり初期値で定まる. したがって $(1, 1)$ 成分の振舞は

$$\psi(\mu, \lambda, t) = Ae^{i(\lambda+\Omega t)} P_1^1(\mu) \quad (25)$$

緯度方向に速度 Ω で逆向きに伝播する. この解は回転系で見れば波数 $(1, 1)$ のロスビー波である. 慣性系から見て保存している極をとる剛体回転流を, 回転系から見ると Ω で逆まわりしているように見えるのがこの解である.

$(1, -1)$ 成分は $(1, 1)$ 成分と同様に粘性項, 非線形項の寄与が 0 となる.

$$-\frac{2}{a^2} \frac{d\tilde{\psi}_{1,-1}}{dt} - i \frac{2\Omega}{a^2} \tilde{\psi}_{1,-1} = 0, \quad \tilde{\psi}_1^1 = Ae^{-i\Omega t}. \quad (26)$$

したがって $(1, 1)$ 成分の振舞は

$$\psi(\mu, \lambda, t) = Ae^{-i(\lambda+\Omega t)} P_1^1(\mu) \quad (27)$$

緯度方向に速度 Ω で逆向きに伝播する. 特に回転がなければ時間微分が 0 となり保存する.

文献

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