2 重フーリエ変換

竹広真一

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この文書は, 2 重フーリエ変換の基本的な定式化を行う.

1　2 重フーリエ変換

x 方向に $[x_{min}, x_{max}]$, y 方向に $[y_{min}, y_{max}]$ の領域での周期関数 $g(x, y)$ のフーリエ変換は

$$s_{k,l} = \frac{D_x}{D_y} \int_{x_{min}}^{x_{max}} \int_{y_{min}}^{y_{max}} g(x, y) \exp\left(-i\frac{2\pi}{D_x}kx - i\frac{2\pi}{D_y}ly\right) dxdy \tag{1}$$

ただし $D_x = x_{max} - x_{min}$, $D_y = y_{max} - y_{min}$ である. 切断波数 K, L とする逆変換は

$$g(x, y) = \sum_{k=-K}^{K} \sum_{l=-L}^{L} s_{k,l} \exp\left(i\frac{2\pi}{D_x}kx + i\frac{2\pi}{D_y}ly\right) \tag{2}$$

x, y それぞれ $[0, 2\pi]$ の範囲に変換する. すなわち $x = x_{min} + D_x/(2\pi)\tilde{x}, y = y_{min} + D_y/(2\pi)\tilde{y}$ と変換すると正変換, 逆変換はそれぞれ

$$s_{k,l} = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} g(x, y) \exp(-ikx - ily) dxdy, \tag{3}$$

$$g(x, y) = \sum_{k=-K}^{K} \sum_{l=-L}^{L} s_{k,l} \exp\left(ikx + ily\right). \tag{4}$$

となる. ただし簡単のため $\tilde{x}$ を x,$\tilde{y}$ を y に置き換えた.

$g(x,y)$ が実数であることから $s_{k,l}$ に制約が加わる. 逆変換の式で波数 $(k,l),(-k,-l)$ 成分を書き出すと

$$
\begin{aligned}
& s_{k,l}e^{ikx+ily} + s_{-k,-l}e^{-ikx-ily} \\
= \ & Re[s_{k,l}]\cos(kx+ly) - Im[s_{k,l}]\sin(kx+ly) \\
& +i\{Im[s_{k,l}]\cos(kx+ly) + Re[s_{k,l}]\sin(kx+ly)\} \\
& +Re[s_{-k,-l}]\cos(-kx-ly) - Im[s_{-k,-l}]\sin(-kx-ly) \\
& +i\{Im[s_{-k,-l}]\cos(-kx-ly) + Re[s_{-k,-l}]\sin(-kx-ly)\} \\
= \ & Re[s_{k,l}]\cos(kx+ly) - Im[s_{k,l}]\sin(kx+ly) \\
& +i\{Im[s_{k,l}]\cos(kx+ly) + Re[s_{k,l}]\sin(kx+ly)\} \\
& +Re[s_{-k,-l}]\cos(kxly) + Im[s_{-k,-l}]\sin(kx+ly) \\
& +i\{Im[s_{-k,-l}]\cos(kx+ly) - Re[s_{-k,-l}]\sin(kx+ly)\} \\
= \ & \{Re[s_{k,l}] + Re[s_{-k,-l}]\}\cos(kx+ly) + \{-Im[s_{k,l}] + Im[s_{-k,-l}]\}\sin(kx+ly) \\
& +i\{Im[s_{k,l}] + Im[s_{-k,-l}]\}\cos(kx+ly) + i\{Re[s_{k,l}] - Re[s_{-k,-l}]\}\sin(kx+ly)\}
\end{aligned}
$$

この虚数部が 0 とならねばならない. したがって

$$Re[s_{k,l}] = Re[s_{-k,-l}], Im[s_{k,l}] = -Im[s_{-k,-l}], \quad i.e. \quad s_{k,l} = s^*_{-k,-l} \tag{5}$$

である.

2　スペクトル計算

流線関数 $\psi(x,y)$ で表される場の全運動エネルギーならびにエンストロフィーを計算し, スペクトル成分で書き表す. 全運動エネルギーは

$$E = \frac{1}{2}\int_{x_{min}}^{x_{max}}\int_{y_{min}}^{y_{max}}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial y}\right)^2\right]dxdy \tag{6}$$

部分積分すると

$$
\begin{aligned}
E \ = \ & \int_{y_{min}}^{y_{max}}\left[\psi\frac{\partial\psi}{\partial x}\right]_{x_{min}}^{x_{max}}dy + \int_{x_{min}}^{x_{max}}\left[\psi\frac{\partial\psi}{\partial y}\right]_{y_{min}}^{y_{max}}dx - \frac{1}{2}\int_{x_{min}}^{x_{max}}\int_{y_{min}}^{y_{max}}\psi\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right]dxdy \\
= \ & -\frac{1}{2}\int_{x_{min}}^{x_{max}}\int_{y_{min}}^{y_{max}}\psi\left[\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right]dxdy
\end{aligned}
$$

ここで周期境界条件を用いている. x,y それぞれ $[0,2\pi]$ の範囲に変換すべく $x = x_{min} + D_x/(2\pi)x^*, y = y_{min} + D_y/(2\pi)y^*$ とすると,

$$E \ = \ -\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\psi\left[\left(\frac{2\pi}{D_x}\right)^2\frac{\partial^2\psi}{\partial x^{*2}} + \left(\frac{2\pi}{D_y}\right)^2\frac{\partial^2\psi}{\partial y^{*2}}\right]\frac{D_x}{2\pi}\frac{D_y}{2\pi}dx^*dy^*$$

$$= \quad -\frac{D_x D_y}{(2\pi)^2}\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\psi\left[\left(\frac{2\pi}{D_x}\right)^2\frac{\partial^2\psi}{\partial x^{*2}}+\left(\frac{2\pi}{D_y}\right)^2\frac{\partial^2\psi}{\partial y^{*2}}\right]dx^*dy^*$$

ここで流線関数のスペクトル

$$\psi(x,y)=\sum_{k=-K}^{K}\sum_{l=-L}^{L}\tilde{\psi}\exp\left(i\frac{2\pi}{D_x}kx+i\frac{2\pi}{D_y}ly\right)=\sum_{k=-K}^{K}\sum_{l=-L}^{L}\tilde{\psi}_{kl}\exp\left(ikx^*+ily^*\right) \tag{7}$$

を代入すると,

$$\begin{aligned}
E \quad &= \quad \frac{D_x D_y}{(2\pi)^2}\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\sum_{k=-K}^{K}\sum_{l=-L}^{L}\tilde{\psi}_{k,l}e^{ikx^*+ily^*} \\
&\quad \cdot\sum_{k'=-K}^{K}\sum_{l'=-L}^{L}\left[\left(\frac{2\pi k'}{D_x}\right)^2+\left(\frac{2\pi l'}{D_y}\right)^2\right]\tilde{\psi}_{k'l'}e^{ik'x^*+il'y^*}dx^*dy^* \\
&= \quad \frac{D_x D_y}{(2\pi)^2}\frac{1}{2}\sum_{k=-K}^{K}\sum_{l=-L}^{L}\sum_{k'=-K}^{K}\sum_{l'=-L}^{L}\left[\left(\frac{2\pi k'}{D_x}\right)^2+\left(\frac{2\pi l'}{D_y}\right)^2\right]\tilde{\psi}_{k,l}\tilde{\psi}_{k'l'} \\
&\quad \cdot\int_0^{2\pi}e^{i(k+k')x^*}dx^*\int_0^{2\pi}e^{i(l+l')y^*}dy^* \\
&= \quad \frac{D_x D_y}{(2\pi)^2}\frac{1}{2}\sum_{k=-K}^{K}\sum_{l=-L}^{L}\sum_{k'=-K}^{K}\sum_{l'=-L}^{L}\left[\left(\frac{2\pi k'}{D_x}\right)^2+\left(\frac{2\pi l'}{D_y}\right)^2\right]\tilde{\psi}_{k,l}\tilde{\psi}_{k'l'}(2\pi)^2\delta_{k,-k'}\delta_{l,-l'} \\
&= \quad D_x D_y\frac{1}{2}\sum_{k=-K}^{K}\sum_{l=-L}^{L}\left[\left(\frac{2\pi k}{D_x}\right)^2+\left(\frac{2\pi l}{D_y}\right)^2\right]\tilde{\psi}_{k,l}\tilde{\psi}_{-k,-l}.
\end{aligned}$$

最後に $\tilde{\psi}_{-k,-l}=\tilde{\psi}^*_{k,l}$ の関係を用いて

$$E=D_x D_y\frac{1}{2}\sum_{k=-K}^{K}\sum_{l=-L}^{L}\left[\left(\frac{2\pi k}{D_x}\right)^2+\left(\frac{2\pi l}{D_y}\right)^2\right]|\tilde{\psi}_{k,l}|^2, \tag{8}$$

となる．そこで

$$\frac{1}{2}\left[\left(\frac{2\pi k}{D_x}\right)^2+\left(\frac{2\pi l}{D_y}\right)^2\right]|\tilde{\psi}_{k,l}|^2 \tag{9}$$

をエネルギースペクトルの波数 k,l 成分と呼ぶことができる．

同様にエンストロフィーは

$$\begin{aligned}
Q \quad &= \quad \frac{1}{2}\int_{x_{min}}^{x_{max}}\int_{y_{min}}^{y_{max}}\left[\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right]^2 dxdy \\
&= \quad \frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\left[\left(\frac{2\pi}{D_x}\right)^2\frac{\partial^2\psi}{\partial x^{*2}}+\left(\frac{2\pi}{D_y}\right)^2\frac{\partial^2\psi}{\partial y^{*2}}\right]^2\frac{D_x}{2\pi}\frac{D_y}{2\pi}dx^*dy^*
\end{aligned}$$

$$
\begin{aligned}
&= \frac{D_xD_y}{(2\pi)^2}\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\left[\left(\frac{2\pi}{D_x}\right)^2\frac{\partial^2\psi}{\partial x^{*2}}+\left(\frac{2\pi}{D_y}\right)^2\frac{\partial^2\psi}{\partial y^{*2}}\right]^2 dx^*dy^* \\
&= \frac{D_xD_y}{(2\pi)^2}\frac{1}{2}\int_0^{2\pi}\int_0^{2\pi}\sum_{k=-K}^{K}\sum_{l=-L}^{L}\left[\left(\frac{2\pi k}{D_x}\right)^2+\left(\frac{2\pi l}{D_y}\right)^2\right]\tilde{\psi}_{k,l}e^{ikx^*+ily^*} \\
&\quad \cdot\sum_{k'=-K}^{K}\sum_{l'=-L}^{L}\left[\left(\frac{2\pi k'}{D_x}\right)^2+\left(\frac{2\pi l'}{D_y}\right)^2\right]\tilde{\psi}_{k'l'}e^{ik'x^*+il'y^*}dx^*dy^* \\
&= \frac{D_xD_y}{(2\pi)^2}\frac{1}{2}\sum_{k=-K}^{K}\sum_{l=-L}^{L}\sum_{k'=-K}^{K}\sum_{l'=-L}^{L}\left[\left(\frac{2\pi k}{D_x}\right)^2+\left(\frac{2\pi l}{D_y}\right)^2\right]\left[\left(\frac{2\pi k'}{D_x}\right)^2+\left(\frac{2\pi l'}{D_y}\right)^2\right]\tilde{\psi}_{k,l}\tilde{\psi}_{k'l'} \\
&\quad \cdot\int_0^{2\pi}\int_0^{2\pi}e^{ikx^*+ily^*}e^{ik'x^*+il'y^*}dx^*dy^* \\
&= \frac{D_xD_y}{(2\pi)^2}\frac{1}{2}\sum_{k=-K}^{K}\sum_{l=-L}^{L}\sum_{k'=-K}^{K}\sum_{l'=-L}^{L}\left[\left(\frac{2\pi k}{D_x}\right)^2+\left(\frac{2\pi l}{D_y}\right)^2\right]\left[\left(\frac{2\pi k'}{D_x}\right)^2+\left(\frac{2\pi l'}{D_y}\right)^2\right]\tilde{\psi}_{k,l}\tilde{\psi}_{k'l'} \\
&\quad \cdot(2\pi)^2\delta_{k,-k'}\delta_{l,-l'} \\
&= D_xD_y\frac{1}{2}\sum_{k=-K}^{K}\sum_{l=-L}^{L}\left[\left(\frac{2\pi k}{D_x}\right)^2+\left(\frac{2\pi l}{D_y}\right)^2\right]^2\tilde{\psi}_{k,l}\tilde{\psi}_{-k'-l}
\end{aligned}
$$

したがって,

$$
Q = D_xD_y\frac{1}{2}\sum_{k=-K}^{K}\sum_{l=-L}^{L}\left[\left(\frac{2\pi k}{D_x}\right)^2+\left(\frac{2\pi l}{D_y}\right)^2\right]^2|\tilde{\psi}_{k,l}|^2, \tag{10}
$$

となる. そこで

$$
Q = \frac{1}{2}\left[\left(\frac{2\pi k}{D_x}\right)^2+\left(\frac{2\pi l}{D_y}\right)^2\right]^2|\tilde{\psi}_{k,l}|^2, \tag{11}
$$

をエンストロフィースペクトルの波数 k,l 成分と呼ぶことができる.