Magnetoconvection

In lecture 8, we considered how the flow of a conducting fluid affects the magnetic field. Now we will study how the magnetic field affects the flow. We will extend our linear stability analysis for rotating convection in a plane layer (lecture 6) by adding a magnetic field. A Lorentz force must be added to the Navier Stokes equation and the induction equation must be solved along with the other equations.

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + 2\rho\Omega\,\mathbf{e}_z\times\mathbf{u} + \nabla p = \eta\,\nabla^2\mathbf{u} + \rho\,\mathbf{g}' + \mu_o^{-1}(\nabla\times\mathbf{B})\times\mathbf{B}$$

Inertia Coriolis pressure viscosity mod. gravity Lorentz

Scaling length, time, etc. as in 3.3 and using a characteristic magnetic field strength B_o :

$$\frac{1}{\Pr}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \frac{2}{E}\,e_z\times\mathbf{u} + \nabla p = \nabla^2\mathbf{u} + Ra\,T\,\mathbf{e}_z + \frac{Ch\Pr}{Pm}(\nabla\times\mathbf{B})\times\mathbf{B}$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \nabla^2 T \qquad \frac{\partial \mathbf{B}}{\partial t} = \nabla\times(\mathbf{v}\times\mathbf{B}) + \frac{\Pr}{Pm}\nabla^2\mathbf{B} \qquad \nabla\cdot\mathbf{u}=0 \qquad \nabla\cdot\mathbf{B}=0$$

$$Ch = \frac{B_o^2 D^2}{\mu_o\rho\nu\lambda}$$ Chandrasekhar number

$$Pm = \frac{\nu}{\lambda}$$ magnetic Prandtl number

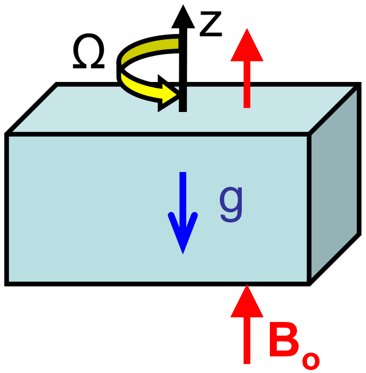
The Chandrasekhar number is a measure for the relative importance of Lorentz forces to viscous forces. The ratio q = Pm/Pr is sometimes called the Roberts number. We abbreviate Ch‘ = ChPr/Pm

Symbols (index R - rotating reference frame, I – inertial frame): Ω – rotation frequency, g‘ – modified gravity, U – characteristic

Magnetoconvection: Linear stability

Solving only the induction equation for a given **u** is the kinematic dynamo problem. Solving the full set of equations on 9.1, with boundary conditions for B that imply no external source, is the magnetohydrodynamic dynamo problem (if boundary conditions for u imply no external driving of the flow, it is a convection-driven dynamo). If we solve the full set of equations with a magnetic field **B** imposed by boundary conditions, this is the magnetoconvection problem.

Consider a plane rotating layer of a conducting fluid with an imposed temperature contrast and an imposed magnetic field B_o. The rotation, gravity and magnetic field vector are all parallel in z-direction. Consider the stability of the trivial solution T=1-z, u=0, $\mathbf{B}=B_o\mathbf{e}_z$ by studying the growth or decay of small perturbations. The magnetic field in the fluid is $\mathbf{B}=B_o\mathbf{e}_z+\mathbf{b}(x,y,z)$, with b the perturbation, which we represent by poloidal and toroidal vector potentials $b = \nabla\times(\nabla\times g\mathbf{e}_z) + \nabla\times h\mathbf{e}_z$. Linearized equations (compare 9.6) with the Lorentz term added:



$$\frac{1}{\Pr}\frac{\partial}{\partial t}\zeta = \nabla^2\zeta + \frac{2}{E}\frac{\partial}{\partial z}w + Ch'\frac{\partial j_z}{\partial z};\qquad \frac{1}{\Pr}\frac{\partial}{\partial t}\nabla^2 w = \nabla^4 w + Ra\,\Delta_2\theta - \frac{2}{E}\frac{\partial\zeta}{\partial z} + Ch'\frac{\partial}{\partial z}\nabla^2 b_z$$

We use (as in case of the velocity) $b_z=-\Delta_2 g$ and $j_z=-\Delta_2 h$ to represent the poloidal and toroidal parts of the induced field. Note that the imposed field B_o itself is force-free and the interaction of the currents of the induced field with B_o gives rise to Lorentz forces.

Symbols: B_o – imposed field, b – induced field (perturbation), g,h – poloidal / toroidal magnetic potentials

Magnetoconvection: Linear stability

We need two more equations for b_z and j_z, which we obtain from the linearized induction equation: $\partial \mathbf{b}/\partial t = \nabla\times(\mathbf{u}\times\mathbf{e}_z) + Pr/Pm\, \nabla^2\mathbf{b}$ or $\partial \mathbf{b}/\partial t = \partial \mathbf{u}/\partial z + Pr/Pm\, \nabla^2\mathbf{b}$. The z-component of this equation and the z-component of the curl of this equation gives:

$$\frac{\partial b_z}{\partial t} = \frac{\partial w}{\partial z} + \frac{\Pr}{Pm}\nabla^2 b_z; \qquad \frac{\partial j_z}{\partial t} = \frac{\partial \zeta}{\partial z} + \frac{\Pr}{Pm}\nabla^2 j_z; \qquad \frac{\partial \theta}{\partial t} = w + \nabla^2\theta$$

Boundary conditions: $j_z=0$ at $z=0,1$. b_z and $\partial b_z/\partial z$ must fit continously at $z=0,1$ to an external continuation of the field that satisfies $\nabla^2 b_z=0$ and decays with distance from the layer. Expanding b_z and j_z into normal modes (compare 6.5):

$b_z(x,y,z) = H_{klm} \exp(ik_x x+ik_y y+\sigma t)\,[\cos(m\pi z) + e^{-K/2} \sinh(K(z-1/2))]$

$j_z(x,y,z) = J_{klm} \exp(ik_x x+ik_y y+\sigma t) \sin(m\pi z)$

The function for the z-dependence of b_z is more complex, in order to satisfy the non-local boundary condition. The part with the sinh-term satisfies $\nabla^2 b_z=0$ inside the layer. Insert into the linearized induction equations for the critical state ($\sigma=0$):

$0 = \pi W - Pr/Pm\,(K^2+\pi^2)\, H \quad \Rightarrow \quad H = W\, Pr/Pm\, \pi/(K^2+\pi^2)$

$0 = -\pi Z - Pr/Pm\,(K^2+\pi^2)\, J \quad \Rightarrow \quad J = Z\, Pr/Pm\, \pi/(K^2+\pi^2)$

The poloidal flow induces a poloidal magnetic field and the toroidal flow induces a toroidal magnetic field. At the critical state, the magnetic field components are a simple linear functions of the respective flow components.

Symbols: H – ampitude of poloidal magnetic modes, J –amplitude of toroidal magnetic modes

Magnetoconvection without rotation

. Set $E \to \infty$. There is no source term in the equation for ζ on 9.2 because j_z is directly related to ζ $\Rightarrow$ no toroidal flow and no toroidal magnetic field are created. Insert the expression for H (9.3) to eliminate H in the modal form of the equation for w (9.2) [the dependence on Pr/Pm drops out and Ch' is replaced by Ch). Calculate the critical Rayleigh number as in lectures 3 and 6 (for m=1):

$$Ra_c(K) = \frac{(K^2+\pi^2)^3}{K^2} + \frac{Ch\,\pi^2\,(K^2+\pi^2)}{K^2}$$

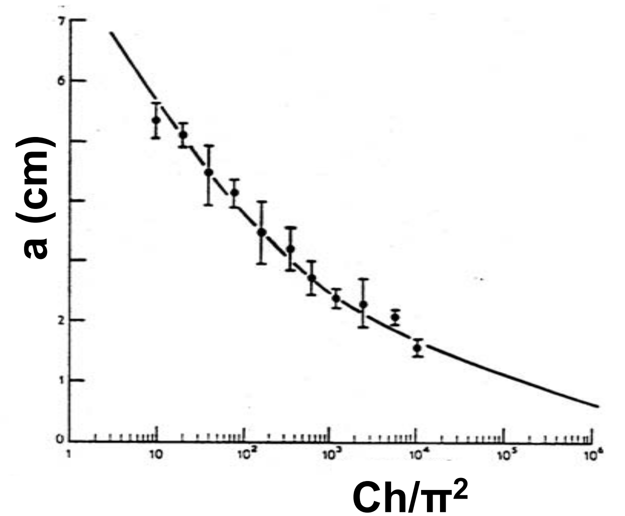
Finding the minimum of Ra_c with respect to K gives finally Ra_{crit}. The relevant limit is Ch >> 1 (Lorentz forces much stronger than viscous forces). In this case, we obtain:

$$Ra_{crit} = \pi^2\,Ch \qquad K_{crit} = (Ch\,\pi^4/2)^{1/6} \approx 1.9\,Ch^{1/6}$$

1) Similarly as rotation, the presence of a magnetic field parallel to g inhibits convection. This can be observed at sunspots, which are strong magnetic flux tubes penetrating through the solar surface. The suppression of convective heat transport makes them cooler and darker.
2) In laboratory experiment $Ch \approx 10^5$ and th the Earth's core $Ch \approx 10^{13}$.

Discussion of linear stability results

Experiments with liquid mercury confirm the theoretical dependence of the wavenumber (aspect ratio a) on Ch.



3) The condition of criticality can be written as Ra $Ch^{-1} > \pi^2$, or in terms of physical parameters: $\frac{\rho\alpha\, g\, \Delta T\, D\, \lambda \mu_o}{\kappa B_o^2} > \pi^2$

Viscosity drops out and is replaced by B_o as factor inhibiting convection.

4) We estimate the relative amount of viscous dissipation of energy, $D_\nu = \rho\nu\, \mathbf{u}\cdot\nabla^2\mathbf{u}$ and ohmic dissipation $D_\lambda = 1/(\mu_o\lambda)\, \mathbf{j}\cdot\mathbf{j} = 1/(\mu_o^2\lambda)\, (\nabla\times\mathbf{b})^2$.

$u \sim \kappa W/D$, $\nabla^2\mathbf{u} \sim \kappa\, KW/D^3$, $D_v \sim \rho\nu\kappa^2\, K^2W^2/D^4$

$\nabla\times\mathbf{b} \sim B_oKH/D \sim B_o\lambda W/(\kappa K D)$, $D_\lambda \sim Ch\, \rho\nu\kappa^2\, K^{-2}W^2/D^4$

The ratio is $D_\lambda/D_v \sim Ch\, K^{-4}$ and with $K_{crit} \sim Ch^{1/6}$: $\Rightarrow$ $\mathbf{D_\lambda/D_v \sim Ch^{1/3}}$

At large Ch, the dissipation of energy is almost entirely ohmic. The additional sink of energy can be considered as reason for the inhibiting influence of B_o.

5) When $B_o \perp g$, convection occurs in the form of rolls aligned with the direction of B. No currents are induced and the critical Rayleigh number is not affected by B.

Magnetoconvection with rotation

. Assume finite E and Ch. The full set of equations in 9.2 must be considered in modal form. The magnetic field can be eliminated by using the relations to the velocity modes given in 9.3.

$$Ra_c(K) = \frac{(K^2+\pi^2)^3}{K^2} + \frac{4\pi^2}{E}\frac{1}{K^2}\frac{(K^2+\pi^2)^2}{(K^2+\pi^2)^2 + Ch\pi^2} + \frac{Ch\,\pi^2\,(K^2+\pi^2)}{K^2}$$

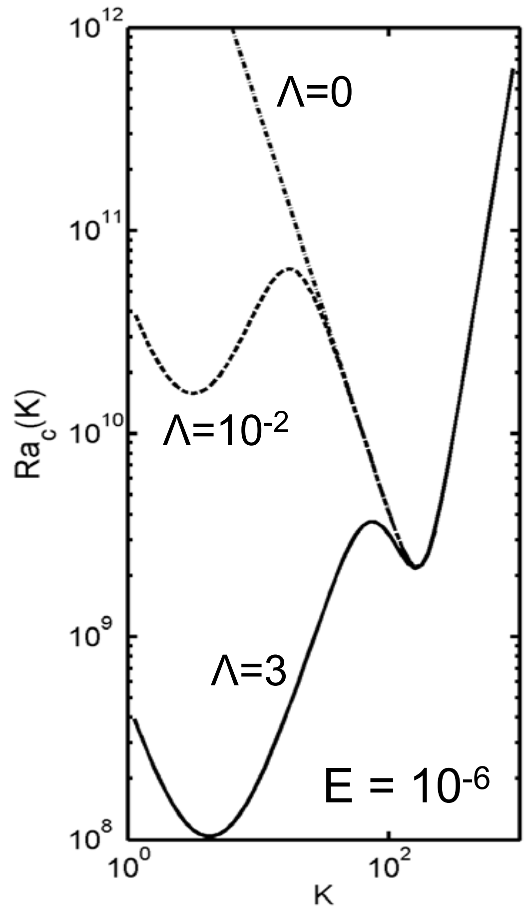
This equation combines the contributions from the simple Bénard problem (viscosity), rotation, and magnetic field. However, the second term related to rotation is multiplied by a „correction factor“ depending on the magnetic field. It stems from the effects of the toroidal magnetic field j_z that is generated by the toroidal flow which is excited by the action of the Coriolis force.
The product Λ = Ch E is given a special name.

$$\Lambda = \frac{B_o^2}{\rho\mu_o\lambda\Omega}$$

Elsasser number ~ Lorentz / Coriolis force

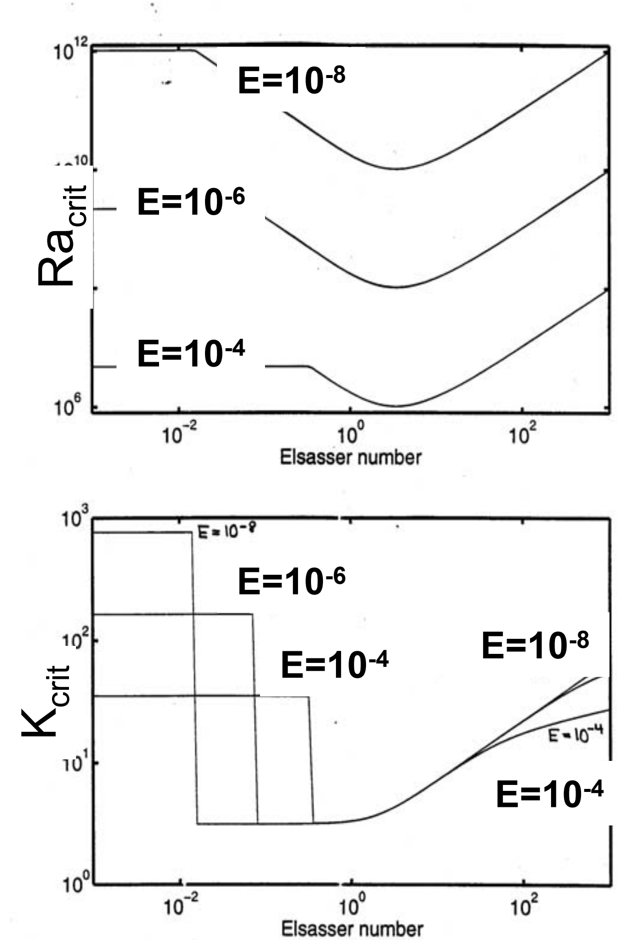
At E << 1 and Ch >> 1, the function $Ra_c(K)$ has two minima. Which is the absolute mimimum, and therefore provides Ra_{crit}, depends on Λ.

Magnetoconvection with rotation: results



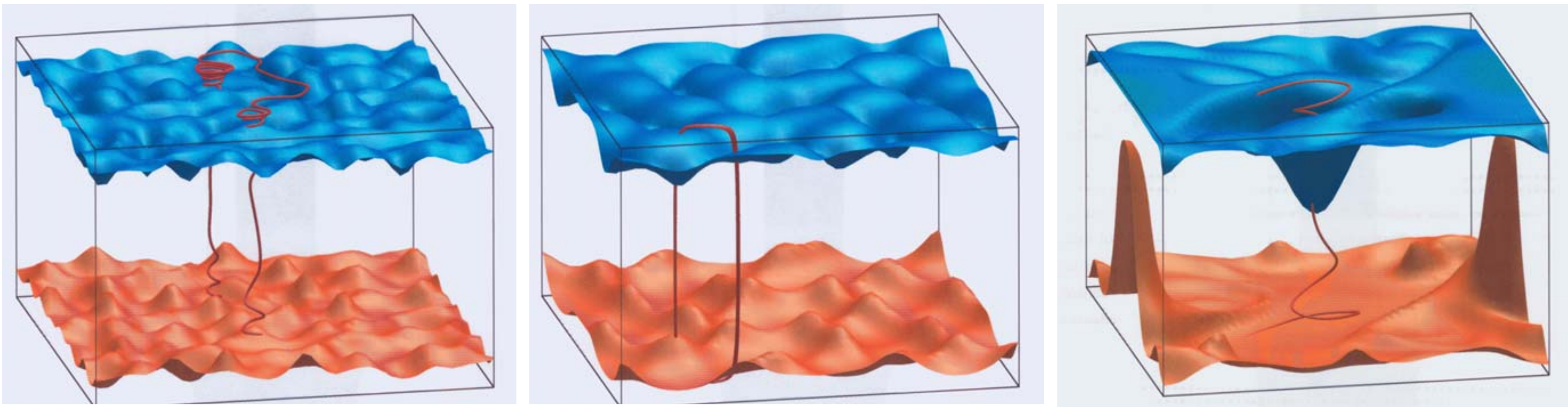
Ra_c as function of K for different values of Λ at E=10^{-6}.

For sufficiently large Λ the solution jumps from large K (narrow cells) to small K. For E→0 the optimum value is $\Lambda=12^{1/2}\approx3.5$ $\Rightarrow K_{crit}=2^{1/2}\pi$ (a≈0.7) and $Ra_{crit} \approx 103\ E^{-1}$. Compare to variation $\sim E^{-4/3}$ without magnetic field. The reduction of Ra_{crit} at the optimum value of Λ is by a factor 20 at E=10^{-6} and by a factor 20,000 at E=10^{-15}.



Critical Rayleigh number and critical wavenumber as function of Λ for various values of E

Comparison rotation/magnetic convection



$E = 10^{-4}$ Ch=0 E=∞ Ch=10^4 E=10^{-4} Ch=10^4 (Λ=1)

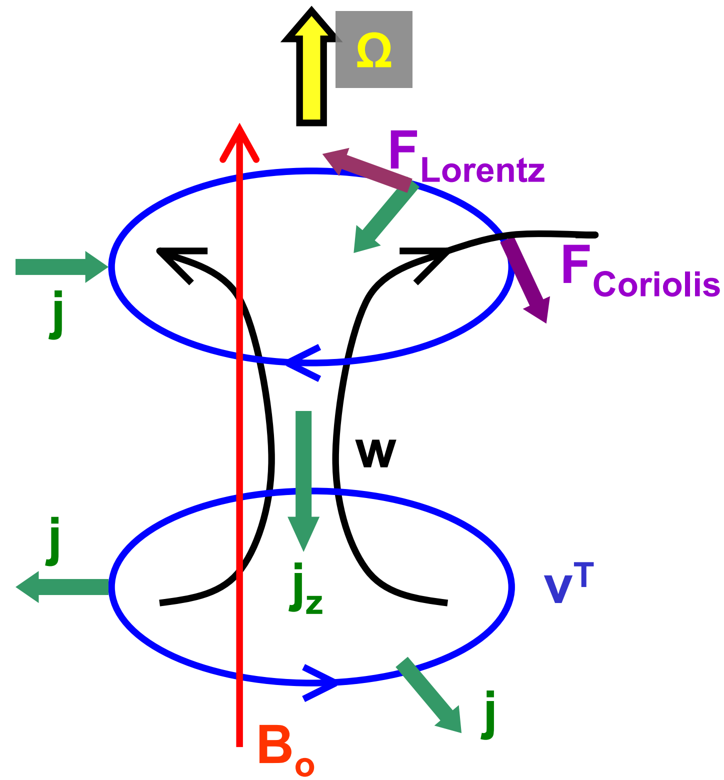
Convection with rotation and/or magnetic field at slightly supercritical Rayleigh number. Two isosurfaces of temperature and the path of a particle moving with the flow are shown.

Rotation alone and magnetic field alone impede convection. Paradoxically, the addition of a magnetic field in case of rapid rotation, such that Lorentz force ≈ Coriolis force (Λ=O(1)), favors convection. The width of the convection cells increases from O($E^{1/3}$) to O(1) and the critical Rayleigh number is reduced by a factor $E^{-1/3}$.

Force balance in magnetorotational convection

Why does the addition of the magnetic field help rotational convection?

Consider an upwelling plume (flow lines in black). The plume flow converges at the bottom and diverges at the top and the action of the Coriolis force (pink arrow) deflects it to the right, creating the toroidal vortices (in blue). Without magnetic force, the toroidal flow is $v^T \sim Z/K \approx 2\pi E^{-1} K^{-3} W$ (6.5) and for small $K \approx \pi$ and $E \ll 1$ $v^T \gg W$. The toroidal flow transports no heat, but only dissipates energy ($D_v \sim v^T K^2 \sim E^{-1} K^{-1} W$). To reduce D_v, then flow must assume a large K (until dissipation by the poloidal flow $\sim WK^2$ becomes comparable).

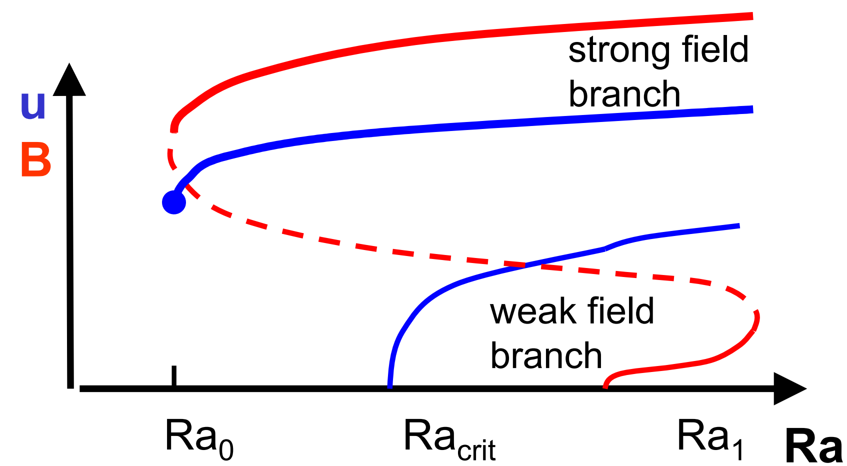


When a magnetic field is present, the toroidal flow induces an electric current (in green) $\mathbf{j} \sim \mathbf{v}^T \times \mathbf{B}_o$, flowing inside the upper vortex and outside the lower vortex (closing by a vertical current j_z). The associated Lorentz force $\mathbf{j} \times \mathbf{B}_o$ is opposed to Coriolis force and for $\Lambda \sim O(1)$ almost cancels it. Therefore $v^T \approx W$ even for small K, which implies much less viscous dissipation.

In the presence of a magnetic field, Lorentz forces can replace viscous forces to „beat“ the Proudman-Taylor theorem.

Magnetoconvection and dynamos

1) The magnetic field has a similar effect also for other orientations between $\mathbf{B}_o$, $\mathbf{\Omega}$ and $\mathbf{g}$.
2) The magnetic field has a similar effect in case of spherical shell convection.
3) The poloidal magnetic field at the surface of the core is ≈ 0.4 mT. Inside the core we can estimate that it is 3-10 times stronger (some even assume very strong toroidal fields up to 50 mT), which implies $\Lambda \approx 1 - 10$ (1000). One possible reason why the Earth's magnetic field has a strength corresponding to $\Lambda = O(1)$ is that convection is most efficient in this case.
4) The dynamo onset in rapidly rotating convection of a conductor may be complex: At Ra_{crit} convection starts, but u is too slow (Rm too small) to generate a magnetic field. At Ra_1 the dynamo starts to generate a weak field ($\Lambda \ll 1$). At some point B becomes strong enough to affect the flow (liberating it from the P-T-constraint) and runaway growth of B and u occur until $\Lambda \approx 1$ is reached.



When initially a strong field is present, the dynamo starts at $Ra_o < Ra_{crit}$ (subcritical onset). Subcritical dynamos have been found numerically in plane-layer geometry, but not so far in a sphere. It remains an open question to what extent magnetoconvection results (with homogeneous B) can be applied to dynamos (with a complex B).