Fundamentals of thermal convection I

Navier-Stokes-Equation (conservation of momentum)

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \nabla p = \eta\,\nabla^2\mathbf{u} + \mathbf{F}_{ext}$$

Inertial — pressure — viscous — external force

incompressible flow

$$\nabla\cdot\mathbf{u} = 0$$

(conservation of mass)

Assumptions and approximations: incompressible flow (ρ=const), inertial (non-rotating) frame of reference, constant Newtonian viscosity. External forces considered are gravitational forces and electromagnetic forces.

Boundary conditions:	(1)	$\mathbf{u} = 0$	impenetrable no-slip boundary
	(2)	$u_n = \partial \mathbf{u}_{\parallel}/\partial n = 0$	impenetrable free-slip boundary

Symbols (bold symbols denote vector): ρ – density, u – velocity, p – pressure, η – dynamic viscosity, n – direction normal to boundary, ıı – direction parallel to boundary

Boussinesq approximation

In thermal convection the flow is driven by differences in temperature that lead by thermal expansion to (usually small) differences in fluid density: $\rho = \rho_o(1 - \alpha T)$. Conflict with assumption of incompressibility. Boussinesq approximation: assume ρ=const in all terms, except in that for the external gravity force: $\mathbf{F}_{ext} = \rho\mathbf{g} = \rho_o\mathbf{g}(1-\alpha T)$. **g** given by gradient of potential: define hydrostatic pressure $\nabla p_H = \rho_o\mathbf{g}$ and dynamic pressure $P = p - p_H$.

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla P/\rho_o \; = \; \nu\,\nabla^2\mathbf{u} - \mathbf{g}\alpha T$$

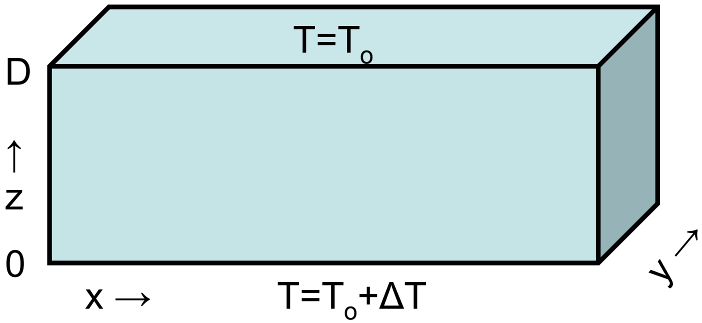
Energy equation (heat transport equation). In the Boussinesq case, adiabatic heating and frictonal heat are zero.

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T \; = \; \kappa\,\nabla^2 T + H'$$

Symbols (index o denotes standard or reference value): α – volumetric thermal expansion coefficient, T – temperature, g – gravity, P – dynamic pressure, $\nu=\eta/\rho$ – kinematic viscosity, κ – thermal diffusivity, $H'=H/(\rho c_p)$ – H is specific heat generation rate per unit volume and c_p is specific heat capacity.

Rayleigh – Bénard convection

Plane layer of height D and large (infinite) horizontal extent filled with a Newtonian fluid with constant material properties. Cartesian coordinates x,y,z, $\mathbf{g} = -g\,\mathbf{e}_z$. Temperature fixed to $T=T_o+\Delta T$ at z=0 and $T=T_o$ at z=D, H'=0.



Scaling of equations

Non-dimensional variables: $(x',y',z') = (x,y,z)/D$, $t' = t\,\kappa/D^2$, $T'=(T-T_o)/\Delta T$, $\mathbf{u}' = \mathbf{u}\,D/\kappa$, $P' = P\,D^2/(\kappa\eta)$.

Non-dimensional equations (omitting primes):

$$\frac{1}{\mathrm{Pr}}\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) + \nabla P = \nabla^2\mathbf{u} + Ra\,T\,\mathbf{e}_z$$

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = \nabla^2 T \qquad \nabla\cdot\mathbf{u} = 0$$

$$Ra = \frac{\alpha g \Delta T D^3}{\kappa\nu} \qquad \mathrm{Pr} = \frac{\nu}{\kappa}$$

Rayleigh number   Prandtl number

Boundary conditions: $w = \partial u/\partial z = \partial v/\partial z = 0$ at z=0 and z=1; T(z=0) = 1; T(z=1) = 0.

By scaling, we replace seven physical parameters $(\alpha,\kappa,\nu,\Delta T,T_o,g,D)$ by two numbers.

Symbols: $\mathbf{e}_z$ – unit vector in z-direction (vertical), D – height of layer, Ra – Rayleigh number, Pr – Prandtl number, **u** = (u,v,w) – cartesian velocity components

Linear stability analysis (I)

- Trivial solution: $T = 1 - z$, $\mathbf{u} = 0$, $P = 0$. Is this solution stable, i.e. will small perturbation decay?
- In Earth's mantle Pr >>1, assume Pr = ∞.
- Assume 2-D solution (independence of y, v=0).
- Take curl of Navier-Stokes equation (eliminates pressure). Note that $\nabla\times(T\mathbf{e}_z) = -\partial T/\partial x\ \mathbf{e}_y$.
- Represent 2D incompress. flow by stream function ψ: $\mathbf{u} = (u,0,w) = \nabla\times(\psi\mathbf{e}_y) = (\partial\psi/\partial z, 0, -\partial\psi/\partial x)$.
- Note that for any $\mathbf{a}$ with $\nabla\cdot\mathbf{a}=0$, $\nabla^2\mathbf{a} = -\nabla\times(\nabla\times\mathbf{a})$. Operators ∇^2 and $\nabla\times$ commute. $\boldsymbol{\omega}=\nabla\times\mathbf{u}$ is called vorticity.
- Perturbation $T = 1-z+\theta$, $\theta \ll 1$, $u \ll 1$. Ignore quadratic terms in small quantities.

$$\nabla^4\psi = Ra\,\frac{\partial\theta}{\partial x} \qquad\qquad \frac{\partial\theta}{\partial t} + \frac{\partial\psi}{\partial x} = \nabla^2\theta$$

- Boundary conditions for ψ and θ: $\psi = \partial^2\psi/\partial z^2 = \theta = 0$ at $z=0$ and $z=1$.
- Expand into normal modes in x-direction: $\theta = \theta_k(z,t)\exp(ikx)$; $\psi = \psi_k(z,t)\exp(ikx)$.
- Expand in harmonic function in z-direction: $\theta_k(z,t)=\theta_{kn}(t)\sin(n\pi z)$, $\psi_k(z,t)=\psi_{kn}(t)\sin(n\pi z)$. Note that sine-functions satisfy all the boundary conditions.

Symbols: ψ – stream function, θ – temperature perturbation, k – horizontal wave number

Linear stability analysis (II)

$(k^2+n^2\pi^2)^2\ \psi_{kn} = Ra\ ik\ \theta_{kn}$ $\qquad d\theta_{kn}/dt = -ik\ \psi_{kn} - (k^2+n^2\pi^2)\ \theta_{kn}$ $\qquad$ Eliminate ψ_{kn}:

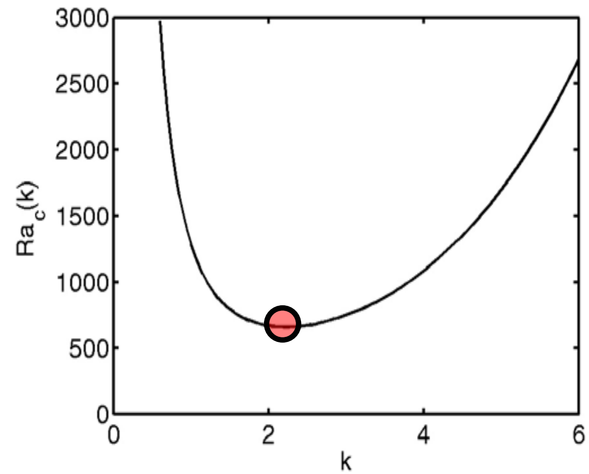
$$\frac{d\theta_{kn}}{dt} = \left(k^2+n^2\pi^2\right)\left(\frac{k^2\,Ra}{(k^2+n^2\pi^2)^3} - 1\right)\theta_{kn} \qquad = \sigma\,\theta_{kn}$$

The solution has the form $\theta_{kn} \sim e^{\sigma t}$. When, at a given value of Ra, $\sigma < 0$ for all n and all k, the trivial (conductive) solution is stable. When for some k and n $\sigma > 0$, the conductive solution is unstable and convection will start. The critical Rayleigh number Ra_{crit} is found by seeking the lowest Ra for which $\sigma=0$ is reached at any k,n. Obviously, the minimum is obtained for n=1.

$Ra_c(k) = (k^2+\pi^2)^3 / k^2$

Minimum at $k_{crit}=\pi/\sqrt{2}$. Wavelength is $\lambda = 2\sqrt{2}$.
Width of one convection cell (aspect ratio) is $\sqrt{2}$.

$Ra_{crit} = Ra_c(k_{crit}) = 27\pi^4/4 \approx 657.5$



Symbols: σ - growth rate, Ra_{crit} - critical Rayleigh number, k_{crit} - critical wave number

Linear stability analysis (III)

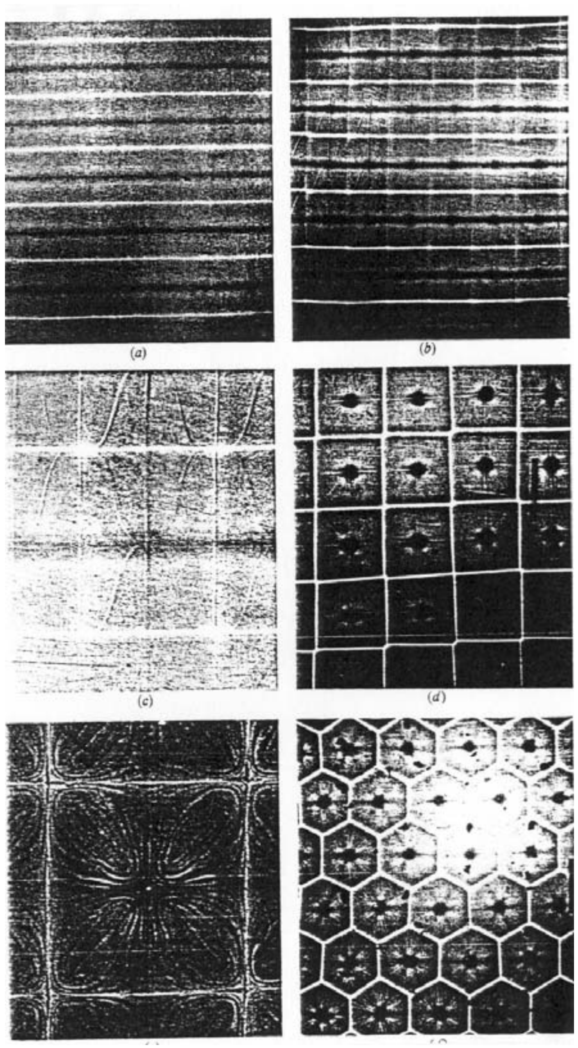
- Result unchanged when 3-D convection pattern is allowed. The critical wave number is then $|k| = (k_x^2+k_y^2)^{1/2} = 2\sqrt{2}$. Linear stability cannot discriminate between different planforms of convection, which is controlled by the non-linear terms. At small super-critical Rayleigh number, the preferred pattern is two-dimensional (convection rolls).
- Result is unchanged if a finite value of Pr is retained.
- Similar analysis (although more complicated) for other boundary conditions. For example with no slip boundaries $Ra_{crit} = 1708$.
- In the case of internal heating, $H > 0$, $\partial T/\partial z=0$ at $z=0$, $T=0$ at $z=1$, the Rayleigh number must be re-defined, replacing ΔT by the characteristic temperature contrast in the conductive state, HD^2/k

$$Ra_H = \frac{\alpha g H D^5}{k \kappa \nu}$$

  The critical Rayleigh number in this case is, with free-slip boundaries, $Ra_{crit} = 868$.
- In a broad range of other cases (various combinations of mechanical and thermal boundary conditions, spherical geometry) the critical Rayleigh number is typically of the order 10^3.

Symbols: k_x, k_y – components of wavenumber vector, k – thermal conductivity,

Pattern of convection at Ra > Ra_{crit}



Visualization of pattern in convection experiments by shadowgraph technique. Regular geometrical pattern are observed at moderate values of teh Rayleigh number, up to ≈ 10 Ra_{crit} .

At larger Rayleigh number the flow becomes irregular and time-dependent.

Symbols: k_x, k_y – components of wavenumber vector, k – thermal conductivity,

Application to Earth and Planets

For Earth's mantle select characteristic values:

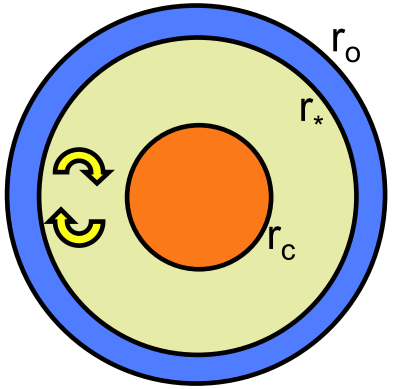
$\alpha = 2 \times 10^{-5}\ K^{-1}$ $\Delta T = 2000\ K$ $g = 10\ m/s^2$ $D = 2{,}900{,}000\ m$

$\kappa = 10^{-6}\ m^2/s$ $\rho = 4000\ kg/m^3$ $\eta = 10^{21} - 10^{22}\ Pa\ s$ (from postglacial rebound)

$$\Rightarrow \quad Ra = 4 \times 10^6 \ \ldots\ 4 \times 10^7 \quad \gg \quad Ra_{crit}$$

For other planets, assume similar values for α, ρ, κ, η. Use Ra_H with the „chondritic" value of radiogenic heating $H = 1.6 \times 10^{-8}\ Wm^{-3}$ ($H' = 4 \times 10^{-15}$ K/s). Without plate tectonics convection takes place below a rigid outer shell whose bottom at radius r_* is given by a temperature of $T_* \approx 1300$ K (heat is conducted in the shell). In a conducting, internally heated sphere $T(r) = H'/(6\kappa)\,[r_o^2 - r^2] + T_o \Rightarrow r_*^2 = r_o^2 - 6\kappa(T_* - T_o)/H'$. Set $D = r_* - r_c$.

	r_o [km]	r_c [km]	T_o [K]	r_* [km]	g_o [m/s²]	Ra_H
Venus	6050	3200	720	5950	8.9	4×10^8
Mars	3400	1500	220	3060	3.7	1×10^7
Moon	1740	400	250	960	1.6	3×10^4



Symbols: r_o – outer radius of planet, T_o – temperature at r_o, T_* - transitional temperature, r_* - radius of lithosphere-asthenosphere boundary, r_c – core radius, g_o – gravity at surface