FDEPS 2012, Lecture 2

Hamiltonian geophysical fluid dynamics

Ted Shepherd

Department of Meteorology

University of Reading

- Hamiltonian dynamics is a very beautiful, and very powerful, mathematical formulation of physical systems
  - All the important models in GFD are Hamiltonian
- Since it is a general formulation, it provides a framework for "meta-theories", providing traceability between different approximate models of a physical system
  - e.g. barotropic to quasi-geostrophic to shallow-water to hydrostatic primitive equations to compressible equations
  - Symmetries and conservation laws are linked by Noether's theorem
- In their pure formulation, Hamiltonian systems are conservative; but the Hamiltonian formulation provides a framework to understand forced-dissipative systems too
  - The nonlinear interactions are generally conservative
  - Example: energy budget (APE and Lorenz energy cycle)
  - Example: momentum transfer by waves

- Hamilton's equations for a canonical system:

$$\frac{\mathrm{d}q_i}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial p_i}, \qquad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial \mathcal{H}}{\partial q_i} \qquad (i = 1, \ldots, N)$$

- For a Newtonian potential system, we get Newton's second law:

$$\mathcal{H} = (|\mathbf{p}|^2/2m) + U(\mathbf{q}) \quad \Rightarrow \quad m\frac{\mathrm{d}^2 q_i}{\mathrm{d}t^2} = -\frac{\partial U}{\partial q_i} \qquad (i = 1, \ldots, N)$$

Conservation of energy follows:
(repeated indices summed)

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial q_i}\frac{\mathrm{d}q_i}{\mathrm{d}t} + \frac{\partial \mathcal{H}}{\partial p_i}\frac{\mathrm{d}p_i}{\mathrm{d}t}$$

$$= \frac{\partial \mathcal{H}}{\partial q_i}\frac{\partial \mathcal{H}}{\partial p_i} - \frac{\partial \mathcal{H}}{\partial p_i}\frac{\partial \mathcal{H}}{\partial q_i} = 0$$

Symplectic formulation:

$$\frac{\mathrm{d}u_i}{\mathrm{d}t} = J_{ij}\frac{\partial \mathcal{H}}{\partial u_j} \qquad (i = 1, \ldots, 2N)$$

$$\mathbf{u} = (q_1, \ldots, q_N, p_1, \ldots, p_N)$$

$$J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}$$

- The symplectic formulation of Hamiltonian dynamics can be generalized to other *J*, which have to satisfy certain mathematical properties
- Among these is skew-symmetry, which guarantees energy conservation:

$$\frac{\mathrm{d}\mathcal{H}}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial u_i}\frac{\mathrm{d}u_i}{\mathrm{d}t} = \frac{\partial \mathcal{H}}{\partial u_i} J_{ij} \frac{\partial \mathcal{H}}{\partial u_j} = 0$$

- The canonical J is invertible. If J is non-invertible, then Casimirs are defined to satisfy

$$J_{ij}\frac{\partial \mathcal{C}}{\partial u_j} = 0 \qquad (i = 1, \ldots, 2N)$$

Casimirs are invariants of the dynamics since

$$\frac{\mathrm{d}\mathcal{C}}{\mathrm{d}t} = \frac{\partial \mathcal{C}}{\partial u_i}\frac{\mathrm{d}u_i}{\mathrm{d}t} = \frac{\partial \mathcal{C}}{\partial u_i} J_{ij} \frac{\partial \mathcal{H}}{\partial u_j} = -\frac{\partial \mathcal{H}}{\partial u_i} J_{ij} \frac{\partial \mathcal{C}}{\partial u_j} = 0$$

- Example of a non-canonical Hamiltonian representation: Euler's equations for a rigid body. The dependent variables are the components of angular momentum about principal axes, and the total angular momentum is a Casimir invariant.
- Cyclic coordinates: e.g. rotational symmetry implies conservation of angular momentum

$$\frac{\partial H}{\partial q_i} = 0 \Rightarrow \frac{dp_i}{dt} = 0 \qquad \text{for a given } i$$

- More generally, the link between symmetries and conservation laws is provided by *Noether's theorem*:

Given a function $\mathcal{F}(\mathbf{u})$, define $\delta_{\mathcal{F}} u_i = \varepsilon J_{ij}(\partial \mathcal{F}/\partial u_j)$

Then $$\delta_{\mathcal{F}} \mathcal{H} = \frac{\partial \mathcal{H}}{\partial u_i} \delta_{\mathcal{F}} u_i = \varepsilon \frac{\partial \mathcal{H}}{\partial u_i} J_{ij} \frac{\partial \mathcal{F}}{\partial u_j}$$

- But
$$\frac{\mathrm{d}\mathcal{F}}{\mathrm{d}t} = \frac{\partial \mathcal{F}}{\partial u_i}\frac{\mathrm{d}u_i}{\mathrm{d}t} = \frac{\partial \mathcal{F}}{\partial u_i} J_{ij} \frac{\partial \mathcal{H}}{\partial u_j}$$

  and hence $\delta_{\mathcal{F}}\mathcal{H} = 0 \text{ if and only if } \mathrm{d}\mathcal{F}/\mathrm{d}t = 0$

- Casimir invariants are associated with 'invisible' symmetries since
$$\delta_{\mathcal{C}}\mathbf{u} = 0$$

- Example: rigid body

- **Barotropic dynamics** is a Hamiltonian system

$$\frac{\partial \omega}{\partial t} = -\mathbf{v} \cdot \nabla \omega = -\partial(\psi, \omega)$$

$$\begin{aligned} \delta\mathcal{H} &= \delta \iint \frac{1}{2}|\nabla\psi|^2 \,\mathrm{d}x\,\mathrm{d}y \\ &= \iint \nabla\psi \cdot \delta\nabla\psi \,\mathrm{d}x\,\mathrm{d}y \\ &= \iint \{\nabla \cdot (\psi\delta\nabla\psi) - \psi\delta\omega\} \,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

$$\delta\mathcal{H}/\delta\omega = -\psi$$

(assuming boundary terms vanish)

- Functional derivatives are just the infinite-dimensional analogue of partial derivatives; they can reflect non-local properties
- Barotropic dynamics can be written in symplectic form as:

$$\frac{\partial \omega}{\partial t} = J \frac{\delta \mathcal{H}}{\delta \omega} \quad \text{where} \quad J = -\partial(\omega, \cdot)$$

- The Casimir invariants are:

$$\mathcal{C} = \iint C(\omega)\, \mathrm{d}x\, \mathrm{d}y \quad \text{with} \quad \frac{\delta \mathcal{C}}{\delta \omega} = C'(\omega)$$

and correspond to Lagrangian conservation of vorticity

- Symmetry in *x* and conservation of *x*-momentum:

$$-\varepsilon \frac{\partial \omega}{\partial x} = \delta_{\mathcal{M}} \omega = \varepsilon J \frac{\delta \mathcal{M}}{\delta \omega} = -\varepsilon \partial\left(\omega, \frac{\delta \mathcal{M}}{\delta \omega}\right)$$

$$\delta \mathcal{M}/\delta \omega = y. \qquad \mathcal{M} = \iint y\omega\, \mathrm{d}x\, \mathrm{d}y = \iint y\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) \mathrm{d}x\, \mathrm{d}y$$

Kelvin's impulse

$$= \iint u\, \mathrm{d}x\, \mathrm{d}y$$

(ignoring boundary terms)

- Similarly for y-momentum and angular momentum:

$$\mathcal{M} = -\iint x\omega \,\mathrm{d}x\,\mathrm{d}y = \iint v \,\mathrm{d}x\,\mathrm{d}y$$

$$\mathcal{M} = -\iint \frac{1}{2} r^2 \omega \,\mathrm{d}x\,\mathrm{d}y = \iint \hat{\mathbf{z}} \cdot (\mathbf{r} \times \mathbf{v}) \,\mathrm{d}x\,\mathrm{d}y$$

- **Quasi-geostrophic dynamics** is analogous; e.g. for continuously stratified flow

$$\mathcal{H} = \iiint \frac{\rho_s}{2} \left\{ |\nabla\psi|^2 + \frac{1}{S}\psi_z^2 \right\} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

$$q(x, y, z, t) = \psi_{xx} + \psi_{yy} + \frac{1}{\rho_s}\left(\frac{\rho_s}{S}\psi_z\right)_z + f + \beta y$$

$$\delta\mathcal{H} = \left[\iint \frac{\rho_s}{S} \psi \delta\psi_z \,\mathrm{d}x\,\mathrm{d}y\right]_{z=0}^{z=1} + \iiint \{\nabla \cdot (\rho_s \psi \delta \nabla\psi) - \rho_s \psi \delta q\} \,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

- Now in addition to potential vorticity $q(x,y,z,t)$, we need to consider potential temperature on horizontal boundaries $\psi_z(x,y,t)$ [and possibly also circulation on sidewalls]
- Note that for the QG model these quantities also evolve advectively, like vorticity in barotropic dynamics:

$$\frac{\partial \omega}{\partial t} = -\mathbf{v} \cdot \nabla \omega = -\partial(\psi, \omega)$$

- Analogously, the Casimir invariants and x-momentum are:

$$\mathcal{C} = \iiint \mathrm{C}(q)\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

$$+ \iint \mathrm{C}_0(\psi_z)\, \mathrm{d}x\, \mathrm{d}y\Big|_{z=0} + \iint \mathrm{C}_1(\psi_z)\, \mathrm{d}x\, \mathrm{d}y\Big|_{z=1}$$

$$\mathcal{M} = \iiint \rho_\mathrm{s} y q\, \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$

$$+ \iint \frac{\rho_\mathrm{s}}{S} y \psi_z\, \mathrm{d}x\, \mathrm{d}y\Big|_{z=0} - \iint \frac{\rho_\mathrm{s}}{S} y \psi_z\, \mathrm{d}x\, \mathrm{d}y\Big|_{z=1}$$

- **Rotating shallow-water dynamics:**

$$\frac{\partial \mathbf{v}}{\partial t} + (f\hat{\mathbf{z}} + \nabla \times \mathbf{v}) \times \mathbf{v} + \nabla\left(\frac{1}{2}|\mathbf{v}|^2\right) = -g\nabla h,$$

$$\frac{\partial h}{\partial t} + \nabla \cdot (h\mathbf{v}) = 0 \qquad \mathcal{H} = \iint \frac{1}{2}\{h|\mathbf{v}|^2 + gh^2\}\,\mathrm{d}x\,\mathrm{d}y$$

$$\frac{\delta \mathcal{H}}{\delta \mathbf{v}} = h\mathbf{v}, \qquad \frac{\delta \mathcal{H}}{\delta h} = \frac{1}{2}|\mathbf{v}|^2 + gh$$

$$J = \begin{pmatrix} 0 & q & -\partial_x \\ -q & 0 & -\partial_y \\ -\partial_x & -\partial_y & 0 \end{pmatrix} \qquad q = (f + \hat{\mathbf{z}} \cdot \nabla \times \mathbf{v})/h$$

$$\mathcal{M} = \iint h(u - fy)\,\mathrm{d}x\,\mathrm{d}y \qquad \mathcal{C} = \iint hC(q)\,\mathrm{d}x\,\mathrm{d}y$$

- **Disturbance invariants:** arguably the most powerful application of Hamiltonian geophysical fluid dynamics
- Ambiguities about the energy of a wave…
- Ambiguities about the momentum of a wave…
- If u=U is a steady solution of a Hamiltonian system, then

$$J \frac{\delta \mathcal{H}}{\delta \mathbf{u}}\bigg|_{\mathbf{u}=\mathbf{U}} = 0$$

- For a canonical system, J is invertible so $\delta\mathcal{H}/\delta\mathbf{u} = 0$ at $\mathbf{u} = \mathbf{U}$.
  - Hence the disturbance energy is quadratic
- But for a non-canonical system, this is not true and the disturbance energy is generally linear in the disturbance
  - Not sign-definite
  - Cannot define stability, normal modes, etc.

- **Pseudoenergy:**

$$J\left.\frac{\delta\mathcal{H}}{\delta\mathbf{u}}\right|_{\mathbf{u}=\mathbf{U}} = 0 \quad \text{implies} \quad \left.\frac{\delta\mathcal{H}}{\delta\mathbf{u}}\right|_{\mathbf{u}=\mathbf{U}} = -\left.\frac{\delta\mathcal{C}}{\delta\mathbf{u}}\right|_{\mathbf{u}=\mathbf{U}}$$ for some Casimir *C*

Thus $\delta(\mathcal{H}+\mathcal{C}) = 0 \;\text{ at }\; \mathbf{u}=\mathbf{U}.$

$\mathcal{A} = (\mathcal{H}+\mathcal{C})[\mathbf{u}] - (\mathcal{H}+\mathcal{C})[\mathbf{U}]$ is then both conserved and quadratic in the disturbance

(pseudoenergy)

- Example: **Available potential energy** (APE) for the 3D stratified Boussinesq equations

$$\mathcal{H} = \iint \left\{\frac{1}{2}\rho_s|\mathbf{v}|^2 + \rho g z\right\} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \qquad \frac{\delta\mathcal{H}}{\delta\mathbf{v}} = \rho_s\mathbf{v}, \qquad \frac{\delta\mathcal{H}}{\delta\rho} = gz$$

- Consider disturbances to a resting basic state $\mathbf{v}=0,\ \rho=\rho_0(z).$

$$\mathcal{C} = \iiint C(\rho)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z \quad \text{with} \quad \frac{\delta\mathcal{C}}{\delta\rho} = C'(\rho)$$

$$\left.\frac{\delta\mathcal{H}}{\delta\mathbf{u}}\right|_{\mathbf{u}=\mathbf{U}} = -\left.\frac{\delta\mathcal{C}}{\delta\mathbf{u}}\right|_{\mathbf{u}=\mathbf{U}} \qquad \frac{\delta\mathcal{H}}{\delta\rho} = gz \quad \frac{\delta\mathcal{C}}{\delta\rho} = C'(\rho) \quad C'(\rho_0) = -gz$$

- In the last expression, *z* must be regarded as a function of ρ_0 via $Z(\rho_0(z)) = z$; so $\rho_0(z)$ must be invertible, i.e. monotonic
- *Hence the basic state must be stably stratified*

$$C(\rho) = -\int^{\rho} gZ(\tilde{\rho})\,\mathrm{d}\tilde{\rho} \qquad \mathcal{A} = \iiint \left\{\frac{\rho_s}{2}|\mathbf{v}|^2 + (\rho - \rho_0)gz - \int_{\rho_0}^{\rho} gZ(\tilde{\rho})\,\mathrm{d}\tilde{\rho}\right\} \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z$$

- The last two terms in the expression for *A* can be written

$$-\int_0^{\rho-\rho_0} g[Z(\rho_0+\tilde{\rho}) - Z(\rho_0)]\,\mathrm{d}\tilde{\rho}$$ (positive definite)

- Small-amplitude approximation $-\dfrac{g(\rho-\rho_0)^2}{2(\mathrm{d}\rho_0/\mathrm{d}z)}$ (APE of internal gravity waves)
- *Such an APE can be constructed for any Hamiltonian system*

- **Pseudomomentum:** In a similar manner, if a basic state u=U is independent of x (i.e. is invariant with respect to translation in x), then by Noether's theorem,

$$\partial\mathbf{U}/\partial x = 0 \quad \text{implies} \quad J\frac{\delta\mathcal{M}}{\delta\mathbf{u}}\bigg|_{\mathbf{u}=\mathbf{U}} = 0$$

which implies $\delta(\mathcal{M}+\mathcal{C}) = 0 \text{ at } \mathbf{u} = \mathbf{U}$ for some Casimir C

$$\mathcal{A} = (\mathcal{M}+\mathcal{C})[\mathbf{u}] - (\mathcal{M}+\mathcal{C})[\mathbf{U}]$$

(pseudomomentum)

is then both conserved and quadratic in the disturbance

- Example: **Barotropic flow on the beta-plane**

$$\frac{\delta\mathcal{M}}{\delta q} = y, \qquad \frac{\delta\mathcal{C}}{\delta q} = C'(q)$$

- Consider disturbances to an x-invariant basic state $q_0(y)$

$$\delta(\mathcal{M}+\mathcal{C}) = 0 \text{ at } q = q_0 \quad \text{implies} \quad C'(q_0) = -y$$

- This is analogous to the formula for APE, and similarly,

$$\mathcal{A} = \iint \left\{ - \int_0^{q-q_0} [Y(q_0 + \tilde{q}) - Y(q_0)]\, d\tilde{q} \right\} dx\, dy$$

where $Y(q_0(y)) = y$, which is negative definite for $dq_0/dy > 0$

- Small-amplitude approximation: $-\dfrac{(q - q_0)^2}{2(dq_0/dy)}$

- If q_0 is defined to be the zonal mean, then $q_0 = \overline{q}, \quad q' = q - \overline{q}$

  and the zonal mean of this expression becomes $-\dfrac{\overline{q'^2}}{2\bar{q}_y}$

- Exactly the same form applies to stratified QG flow, where the negative of this quantity is known as the **Eliassen-Palm (E-P) wave activity**

- N.B. The sign of this quantity corresponds to the sign of the intrinsic frequency of Rossby waves (negative if $dq_0/dy > 0$)

- **Relation to wave action:** there is a classical result that under slowly varying (WKB), adiabatic conditions, wave action is conserved (Bretherton & Garrett 1969 PRSA)

$$\frac{\partial \hat{A}}{\partial t} + \nabla \cdot \vec{c}_g \hat{A} = 0 \quad \text{where} \quad \hat{A} = \frac{\hat{E}}{\hat{\omega}}$$

  $\hat{E}$ is the wave energy (always positive definite) and $\hat{\omega}$ is the intrinsic frequency, both measured in the frame of reference moving with the mean flow

- Hence $\operatorname{sgn}(\hat{A}) = \operatorname{sgn}(\hat{\omega})$
- Under WKB conditions, pseudoenergy and pseudomomentum are related to wave action via $\omega \hat{A}, \quad k \hat{A}$ respectively
- However pseudoenergy and pseudomomentum are more general, and extend beyond WKB conditions
  - They require only temporal or zonal symmetry, respectively, in the background state

- **Stability theorems:** Pseudoenergy and pseudomomentum are conserved in time (for conservative dynamics), and are quadratic in the disturbance (for small disturbances), so for normal-mode disturbances we have

$\mathcal{A} = \mathcal{A}_0 \mathrm{e}^{2\sigma t} \quad \sigma\mathcal{A} = 0$ (σ is the real part of the growth rate)

- Then $\mathcal{A} \neq 0$ implies $\sigma = 0$ (normal-mode stability)
- Therefore these conservation laws can provide sufficient conditions for stability/necessary conditions for instability. Indeed, many normal-mode stability theorems (e.g. Pedlosky 1987) result from expressions of the form

$$\sigma \int \{\cdots\} \, \mathrm{d}x = 0$$

where the integral turns out to be just pseudoenergy or pseudomomentum (or some combination of the two)

- **Example: Charney-Stern theorem.** For stratified QG dynamics, with horizontal boundaries, the pseudomomentum is given by

$$\mathcal{A} = \iiint \rho_0 \left\{ -\int_0^{q-Q} [Y(Q+\tilde{q}) - Y(Q)] \mathrm{d}\tilde{q} \right\} \mathrm{d}x\, \mathrm{d}y\, \mathrm{d}z$$
$$+ \iint \left\{ -\int_0^{\lambda_0 - \Lambda_0} [Y_0(\Lambda_0 - \tilde{\lambda}) - Y_0(\Lambda_0)] \mathrm{d}\tilde{\lambda}\, \mathrm{d}x\, \mathrm{d}y \right|_{z=0}$$

  plus another term with the opposite sign at the top boundary. Here $\Lambda=\Psi_z$ is proportional to potential temperature.

- Baroclinic instability requires terms of opposite signs so A=0:

  Eady model: Interior term vanishes, Λ_y<0 at bottom, Λ_y<0 at top

  Charney model: Q_y>0 in interior, Λ_y<0 at bottom

  Phillips model: Q_y<0 in lower levels, Q_y>0 in upper levels

- Barotropic instability: can be considered a special case

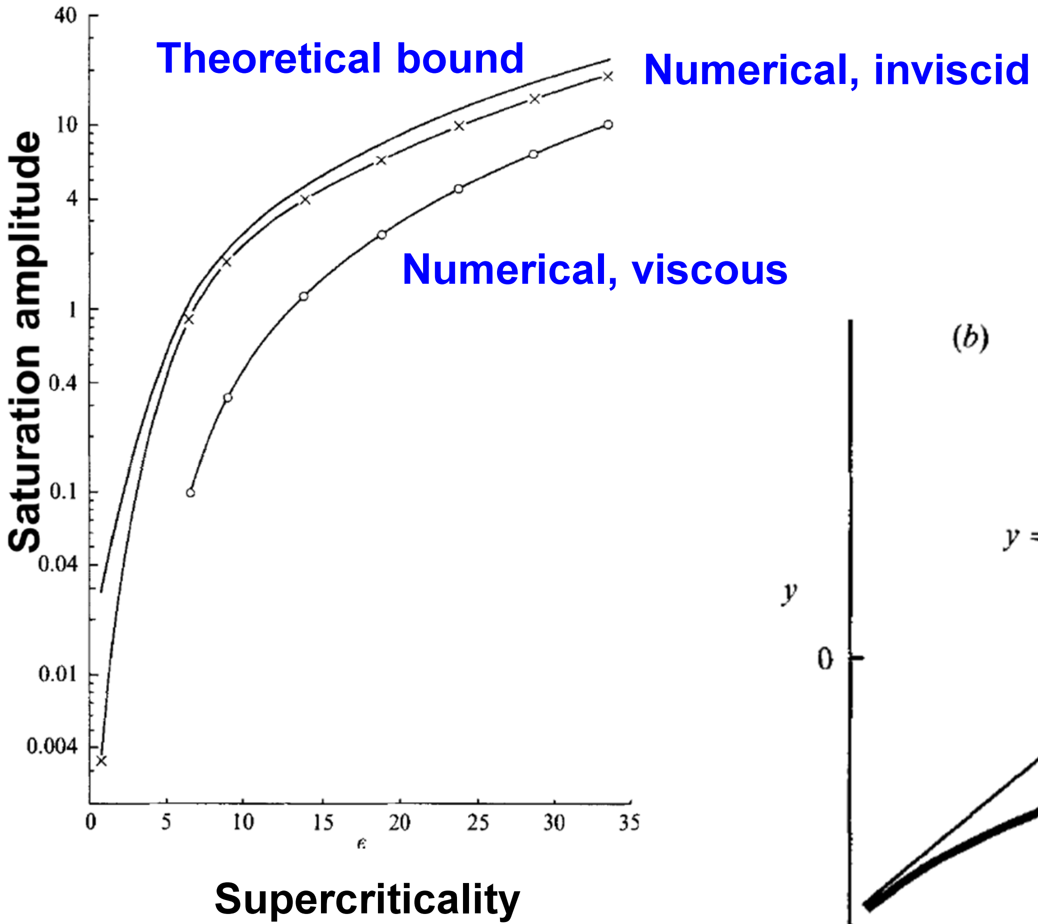
- **Other examples of Hamiltonian stability theorems:**
  - Static stability, centrifugal stability, symmetric stability
  - Rayleigh-Kuo theorem, Fjørtoft-Pedlosky theorem
  - Arnol'd's first and second theorems
  - Ripa's theorem (shallow-water dynamics)
- Notable exception: stratified shear flow (Miles-Howard theorem)
- These Hamiltonian stability theorems can, in most cases, be generalized to finite-amplitude (Liapunov) stability: i.e. for all ε there exists a δ such that

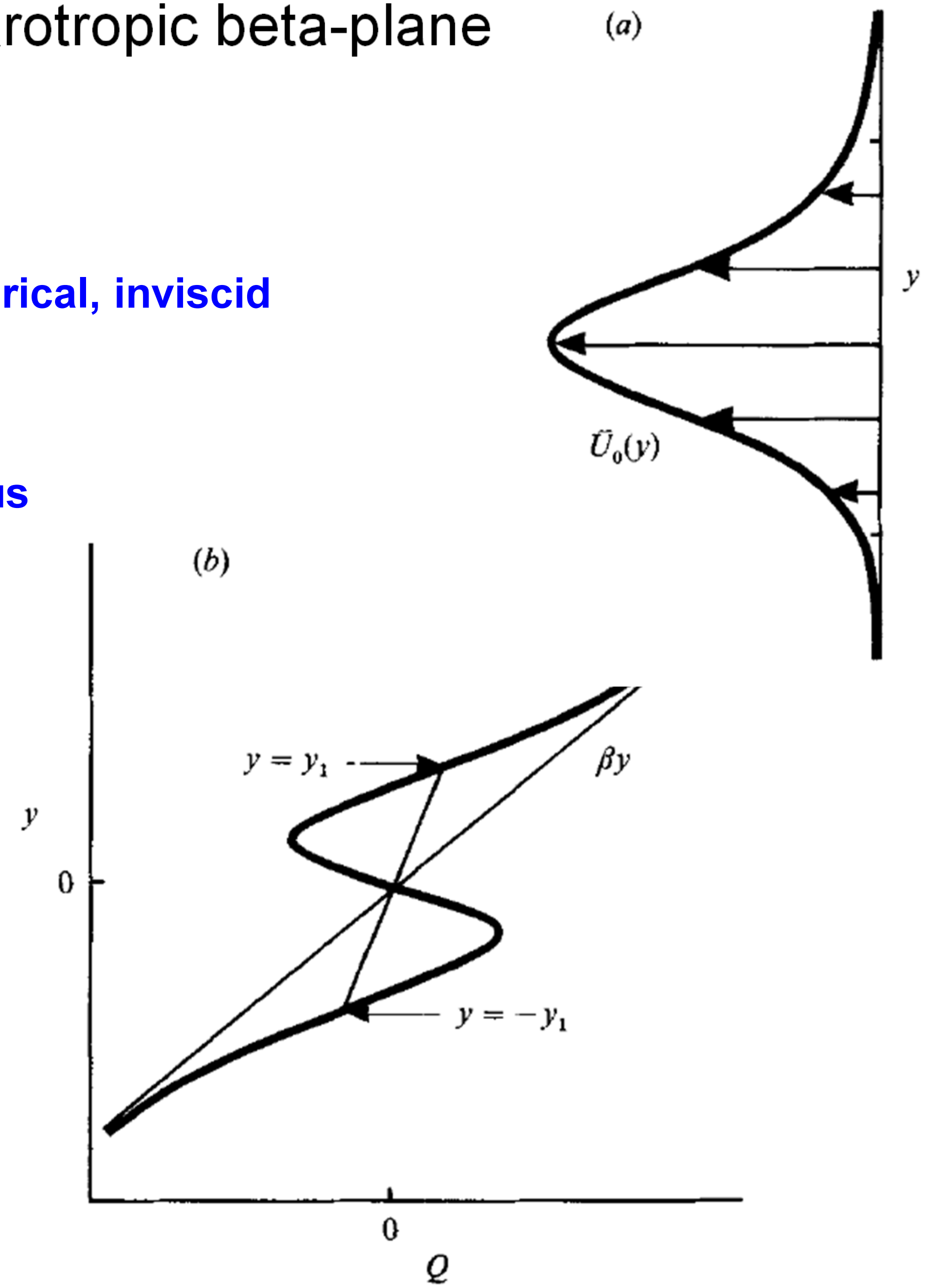
$$\|\boldsymbol{u}'(0)\| < \delta \qquad \Rightarrow \qquad \|\boldsymbol{u}'(t)\| < \epsilon \quad \forall t$$

- They can also be used to derive rigorous saturation bounds on nonlinear instabilities; e.g. for a statically unstable resting state,

$$\iiint \tfrac{1}{2}\rho_0 |\boldsymbol{v}(t)|^2 \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z \le \mathcal{A}(t) = \mathcal{A}(0) = \iiint \mathrm{APE}(0) \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z$$

- **Example:** Bickley jet on barotropic beta-plane (Shepherd 1988 JFM)





- **Relationship between pseudomomentum and momentum:** consider the zonally averaged zonal momentum equation for the barotropic beta-plane:

$$\frac{\partial \bar{u}}{\partial t} = -\overline{\frac{\partial u^2}{\partial x}} - \overline{\frac{\partial uv}{\partial y}} + f\bar{v} - \overline{\frac{\partial p}{\partial x}} = -\frac{\overline{\partial u'v'}}{\partial y}$$

$$\frac{\partial \bar{u}}{\partial t} = \overline{-v'\left(\frac{\partial u'}{\partial y} - \frac{\partial v'}{\partial x}\right)} + \frac{\partial}{\partial x}\left[\tfrac{1}{2}(u'^2 - v'^2)\right] = \overline{v'q'}$$

- The linearized potential-vorticity equation is

$$\frac{\partial q'}{\partial t} + \bar{u}\frac{\partial q'}{\partial x} + v'\frac{\mathrm{d}\bar{q}}{\mathrm{d}y} = 0$$

and hence (if $\bar{q}_y \neq 0$) $\quad v' = -\frac{1}{\bar{q}_y}\left(\frac{\partial q'}{\partial t} + \bar{u}\frac{\partial q'}{\partial x}\right)$

$$\overline{q'v'} = -\frac{\partial}{\partial t}\left(\frac{1}{2}\frac{\overline{q'^2}}{\bar{q}_y}\right) = \frac{\partial \bar{A}}{\partial t} \quad \text{whence} \quad \frac{\partial \bar{u}}{\partial t} = \frac{\partial \bar{A}}{\partial t}$$

(Taylor identity)

- **Stratified QG dynamics:** zonal-wind tendency equation, temperature tendency equation, and thermal-wind balance together imply

$$\mathcal{L}\Big(\frac{\partial \bar{u}}{\partial t}\Big) = \frac{\partial^2}{\partial y^2}\overline{(v'q')} \quad \text{where} \quad \mathcal{L} = \frac{\partial^2}{\partial y^2} + \frac{1}{\rho_0}\frac{\partial}{\partial z}\frac{\rho_0}{S}\frac{\partial}{\partial z}$$

- So it's the same physics, but the zonal-wind response to mixing of potential vorticity is now *spatially non-local* (the Eliassen balanced response): follows from PV inversion

- The pseudomomentum conservation law takes the local form (with *S* being a source/sink)

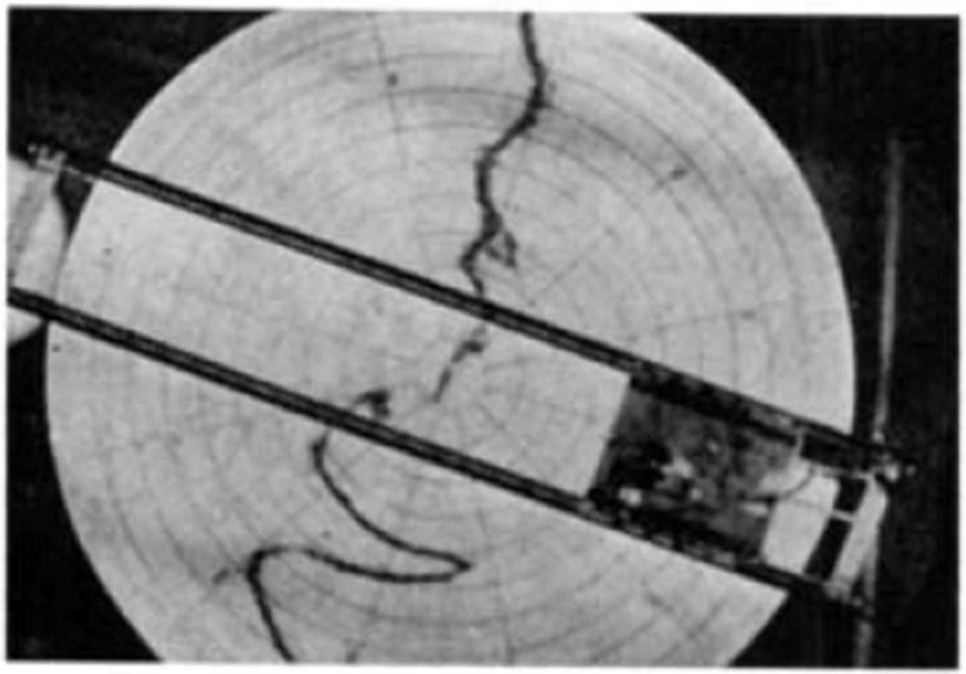
$$\frac{\partial A}{\partial t} + \nabla \cdot \boldsymbol{F} = S \qquad\qquad \nabla \cdot \boldsymbol{F} = -\overline{v'q'}$$

$$\mathcal{L}\Big(\frac{\partial \bar{u}}{\partial t}\Big) = \frac{\partial^2}{\partial y^2}\overline{(v'q')} = -\frac{\partial^2}{\partial y^2}\nabla\cdot\bar{\boldsymbol{F}} = \frac{\partial^2}{\partial y^2}\Big(\frac{\partial \bar{A}}{\partial t} - \bar{S}\Big)$$

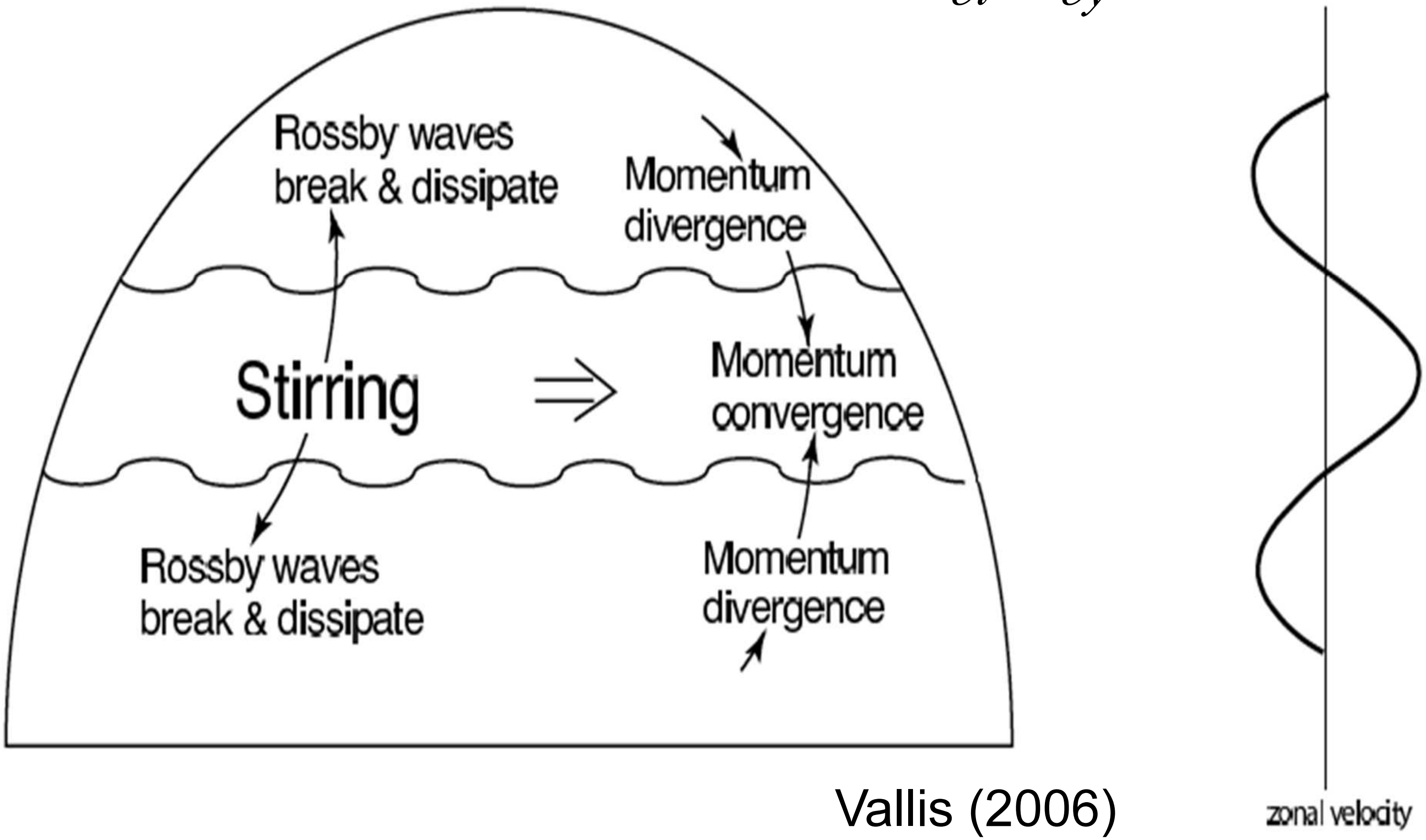
- So mean-flow changes require wave transience or non-conservative effects (*non-acceleration theorem*)

- **In** the atmosphere, we can generally assume that $\bar{q}_y > 0$ since q is dominated by β
- Hence A < 0; *Rossby waves carry negative pseudomomentum*
- Where Rossby waves dissipate, there must be a convergence of negative pseudomomentum, hence a negative torque
- Conservation of momentum implies a positive torque in the wave source region
- This phenomenon is seen in laboratory rotating-tank experiments
- A prograde jet emerges from random stirring, surrounded on either side by retrograde jets (seen in distortion of dye)

(Whitehead 1975 Tellus)



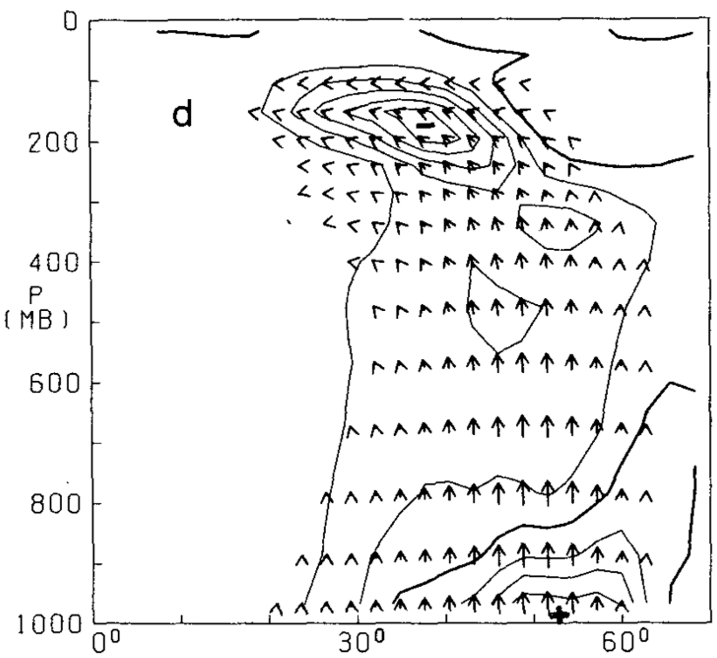
- In the atmosphere, synoptic-scale Rossby waves are generated by baroclinic instability, hence within a jet region
- Flux of negative pseudomomentum out of jet corresponds to an upgradient flux of momentum into the jet $\frac{\partial A}{\partial t}+\frac{\partial}{\partial y}\left(\overline{u'v'}\right)=0, \quad A<0$



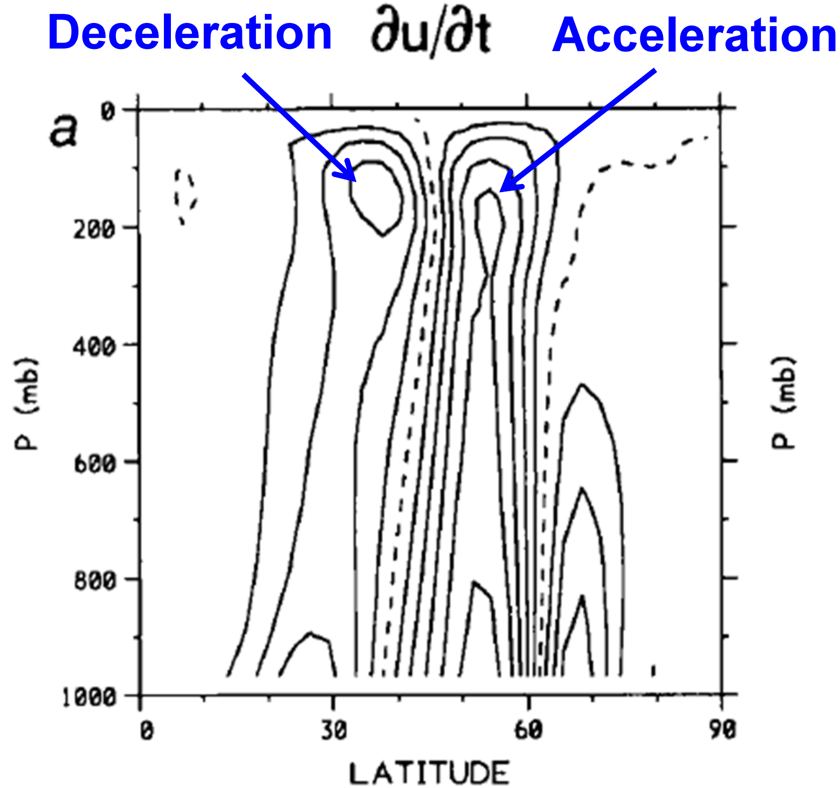
Vallis (2006)

- In fact the wave propagation is up and out (generally equatorward), as seen in these 'baroclinic life cycles' showing baroclinic growth and barotropic decay (Simmons & Hoskins 1978 JAS)

E-P flux (arrows) and divergence (contours)



Edmon, Hoskins & McIntyre (1980 JAS)



Haynes & Shepherd (1989 QJRMS)

- The vertical flux of pseudomomentum is expressed in terms of the meridional heat flux

$$-\boldsymbol{F} = \left(-\rho_0 \overline{u'v'}, (\rho_0 f/\Theta_z)\overline{v'\theta'}\right)$$ is the *Eliassen-Palm (E-P) flux*

- Reflects thermal-wind balance: poleward heat flux weakens the thermal wind, accelerating the flow below and decelerating the flow aloft (as in pure baroclinic instability)
- During the wintertime when the stratospheric flow is westerly, stationary planetary Rossby waves can propagate into the stratosphere where they exert a negative torque, acting to weaken the flow from its radiative equilibrium state
- Stationary planetary-wave forcing mechanisms (topography, land-sea temperature contrast) are stronger in the Northern than in the Southern Hemisphere, hence the stratospheric polar vortex is weaker in the Northern Hemisphere

Summary

- Hamiltonian dynamics is applicable to all the important models of geophysical fluid dynamics
  - Provides a unifying framework between various models
  - Systems are infinite-dimensional, and their Eulerian representations are generally non-canonical
  - To exploit Hamiltonian structure all that is needed is to know the conserved quantities of a system
- The most powerful applications are for theories describing disturbances to an inhomogeneous basic state
  - Non-trivial; e.g., wave energy is generally not conserved
  - Useful measures of disturbance magnitude require the use of Casimir invariants, following from Lagrangian invariants
  - Leads to important concepts of pseudoenergy and pseudomomentum: stability theorems immediately follow
  - Important applications are available potential energy and momentum transfer by waves

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